

A NOTE ON CURVATURE ESTIMATE OF THE HERMITIAN-YANG-MILLS FLOW

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ABSTRACT. In this paper, we study the curvature estimate of the Hermitian-Yang-Mills flow on holomorphic vector bundles. In one simple case, we show that the curvature of the evolved Hermitian metric is uniformly bounded away from the analytic subvariety determined by the Harder-Narasimhan-Seshadri filtration of the holomorphic vector bundle.

1. INTRODUCTION

Let (M, ω) be a compact Kähler manifold and \mathcal{E} be a coherent sheaf on M . A torsion-free coherent sheaf \mathcal{E} is said to be ω -stable (respectively, ω -semistable) in Mumford's sense, if for every coherent proper sub-sheaf $\mathcal{F} \hookrightarrow \mathcal{E}$, it holds:

$$\mu_\omega(\mathcal{F}) = \frac{\deg_\omega(\mathcal{F})}{\text{rank}(\mathcal{F})} < (\leq) \mu_\omega(\mathcal{E}) = \frac{\deg_\omega(\mathcal{E})}{\text{rank}(\mathcal{E})}, \quad (1.1)$$

where $\mu_\omega(\mathcal{F})$ is called the ω -slope of \mathcal{F} , the ω -degree of \mathcal{F} is defined as follow

$$\deg_\omega(\mathcal{F}) = \int_M c_1(\mathcal{F}) \wedge \frac{\omega^{n-1}}{(n-1)!},$$

$c_1(\mathcal{F})$ is the first Chern class of \mathcal{F} .

To an unstable torsion-free coherent sheaf \mathcal{E} , one can associate a filtration by subsheaves

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_k = \mathcal{E}, \quad (1.2)$$

such that the quotients $\mathcal{Q}_i = \mathcal{E}_i / \mathcal{E}_{i-1}$ are torsion-free, ω -semi-stable and $\mu_\omega(\mathcal{Q}_i) > \mu_\omega(\mathcal{Q}_{i+1})$, which is called the Harder-Narasimhan filtration (abbr, HN-filtration) of \mathcal{E} . The associated graded sheaf $Gr^{hn}(\mathcal{E}) = \bigoplus_{i=1}^k \mathcal{Q}_i$ is uniquely determined by the isomorphism class of \mathcal{E} and the Kähler class $[\omega]$. Moreover, for every quotient \mathcal{Q}_i , there is a further filtration by subsheaves

$$0 = \mathcal{E}_{i,0} \subset \mathcal{E}_{i,1} \subset \cdots \subset \mathcal{E}_{i,k_i} = \mathcal{Q}_i, \quad (1.3)$$

such that the quotients $\mathcal{Q}_{i,j} = \mathcal{E}_{i,j} / \mathcal{E}_{i,j-1}$ is torsion-free and ω -stable, $\mu_\omega(\mathcal{Q}_{i,j}) = \mu(\mathcal{Q}_i)$ for each j . This double filtration $\{\mathcal{E}_{i,j}\}$ is called the Harder-Narasimhan-Seshadri filtration (abbr, HNS-filtration) of the sheaf \mathcal{E} . The associated graded sheaf: $Gr^{hns}(E, \bar{\partial}_A, \phi) = \bigoplus_{i=1}^k \bigoplus_{j=1}^{k_i} \mathcal{Q}_{i,j}$ is uniquely determined by the isomorphism class of \mathcal{E} and the Kähler class $[\omega]$. The number $\sum_{i=1}^k k_i - 1$ is called the length of the HNS-filtration.

In the following, we denote $\Sigma_{\mathcal{E}}$ the set of singularities where \mathcal{E} is not locally free. If \mathcal{E} is locally free on the whole M , i.e. $\Sigma_{\mathcal{E}} = \emptyset$, there is a holomorphic vector bundle $(E, \bar{\partial}_E)$ on M such that the sheaf \mathcal{E} is generated by the local holomorphic sections of $(E, \bar{\partial}_E)$. A locally free coherent sheaf \mathcal{E} can be seen as a holomorphic vector bundle, i.e. $\mathcal{E} = (E, \bar{\partial}_E)$. A Hermitian

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metric H on the holomorphic vector bundle $(E, \bar{\partial}_E)$ is said to be ω -Hermitian-Einstein if it satisfies the following Einstein condition on M , i.e.

$$\sqrt{-1}\Lambda_\omega F_H = \lambda_{E,\omega} \text{Id}_\mathcal{E}, \quad (1.4)$$

where $\lambda_{E,\omega} = \frac{2\pi}{\text{Vol}(M,\omega)}\mu_\omega(\mathcal{E})$, F_H is the curvature tensor of Chern connection D_H with respect to the Hermitian metric H , and Λ_ω denotes the contraction with the Kähler metric ω . The Donaldson-Uhlenbeck-Yau theorem ([38, 14, 15, 45]) states that the ω -stability of \mathcal{E} implies the existence of ω -Hermitian-Einstein metric on \mathcal{E} . This theorem has several interesting and important generalizations and extensions ([31, 21, 41, 4, 7, 18, 5, 22, 2, 3, 8, 25, 26, 27, 37, 35, 36], etc.).

Let H_0 be a Hermitian metric on the complex vector bundle E , \mathcal{A}_{H_0} be the space of connections of E compatible with the metric H_0 , and $\mathcal{A}_{H_0}^{1,1}$ be the space of unitary integrable connections of E (i.e. those whose curvature is of type $(1,1)$). For any $A_0 \in \mathcal{A}_{H_0}^{1,1}$, $\bar{\partial}_{A_0} = D_{A_0}^{(0,1)}$ defines a holomorphic structure on E . We consider the following Yang-Mills flow on the Hermitian vector bundle (E, H_0) with initial data A_0 ,

$$\begin{cases} \frac{\partial A}{\partial t} = -D_A^* F_A, \\ A(0) = A_0. \end{cases} \quad (1.5)$$

The Yang-Mills flow, as the gradient flow of the Yang-Mills functional, was first suggested by Atiyah-Bott in [1]. Donaldson [15] proves the global existence of the Yang-Mills flow if the initial data A_0 is integrable. In fact, Donaldson introduces the following Hermitian-Yang-Mills flow for Hermitian metrics $H(t)$ on the holomorphic bundle $(E, \bar{\partial}_{A_0})$ with initial metric H_0 ,

$$\begin{cases} H^{-1} \frac{\partial H}{\partial t} = -2(\sqrt{-1}\Lambda_\omega F_H - \lambda \text{Id}_E), \\ H(0) = H_0, \end{cases} \quad (1.6)$$

and proves that solution to the above nonlinear heat equation exists for all time. Then Donaldson shows that, by choosing complex gauge transformations $\sigma(t)$ which satisfy $\sigma(t)^* \sigma(t) = H_0^{-1} H(t)$, $A(t) = \sigma(t)(A_0)$ is the unique long time solution of the Yang-Mills flow (1.5). Furthermore, Donaldson proves the convergence of the flow at infinity in the case that the initial holomorphic structure $(E, \bar{\partial}_{A_0})$ is stable.

In general case, since the mean curvature tensor $\sqrt{-1}\Lambda_\omega F_{A(t)}$ is uniformly bounded along the Yang-Mills flow, the Uhlenbeck's compactness result ([43, 44]) implies that for any sequence $A(t_i)$ along the flow there is a subsequence, modulo gauge transformations, weakly converges to a Yang-Mills connection A_∞ outside a closed subset Σ_{an} of Hausdorff complex codimension at least 2. We call Σ_{an} the bubbling set. In [23], Hong and Tian have shown that in fact the convergence can be taken to be C_{loc}^∞ on $M \setminus \Sigma_{an}$ and the bubbling set Σ_{an} is a holomorphic subvariety. By Bando and Siu's result ([7]), the holomorphic vector bundle $(E_\infty, \bar{\partial}_{A_\infty})$ on $M \setminus \Sigma_{an}$ can be extended to the whole M as a reflexive coherent sheaf \mathcal{E}_∞ . On the other hand, since A_∞ is Yang-Mills, \mathcal{E}_∞ has a holomorphic orthogonal splitting of stable reflexive sheaves with admissible Hermitian-Einstein metrics. In [7], Bando and Siu conjecture that this limiting reflexive coherent sheaf \mathcal{E}_∞ should be isomorphic to the double dual of the graded object of the Harder-Narasimhan-Seshadri filtration of the initial holomorphic structure $(E, \bar{\partial}_{A_0})$. This isomorphism was pointed out by Atiyah-Bott [1] first for the Riemann surface case, and proved by Daskalopoulos [11]. Later, this conjecture was proved by Daskalopoulos and Wentworth [12] in the case of Kähler surfaces, and Jacob [24] and Sibley [39] for the higher dimension case. This conjecture is also set up in the Higgs bundle case, see references [47, 28, 29].

In the following, we denote Σ_{alg} the singular set of the associated graded sheaf $Gr^{hns}(E, \bar{\partial}_{A_0})$, i.e. $Gr^{hns}(E, \bar{\partial}_{A_0})$ is locally free away from Σ_{alg} . Σ_{alg} is a complex analytic subvariety of complex codimension at least 2, which we call the algebraic singular set. By the results in [39, 24], it is not hard to see that $\Sigma_{alg} \subset \Sigma_{an}$. It is an interesting problem to prove that $\Sigma_{an} \subset \Sigma_{alg}$. This problem was solved by Daskalopoulos and Wentworth [13] for $\dim_{\mathbb{C}} M = 2$, Sibley and Wentworth ([40]) for the higher dimensions case. Sibley and Wentworth's method are mostly algebraic, it should be an interesting problem to give a uniform curvature estimate of the Yang-Mills flow away from the algebraic singular set Σ_{alg} by using analytic methods. In this paper, we solve the problem in the case that $(E, \bar{\partial}_{A_0})$ is nonsemistable and the Harder-Narasimhan-Seshadri filtration is of length one. In fact we obtain the following theorem.

Theorem 1.1. *Let (E, H_0) be a Hermitian vector bundle over a compact Kähler manifold (M, ω) , and $A(t)$ be the solution of the Yang-Mills flow (1.5) with initial connection $A_0 \in \mathcal{A}_{H_0}^{1,1}$. If the holomorphic bundle $\mathcal{E} = (E, \bar{\partial}_{A_0})$ is nonsemistable and the Harder-Narasimhan-Seshadri filtration is of length one. Then, for any compact subset $U \subset M \setminus \Sigma_{alg}$ there exists a uniform constant C_U such that*

$$\sup_{(x,t) \in U \times [0, +\infty)} |F_A|_{H_0}^2(x, t) \leq C_U. \quad (1.7)$$

We believe the theorem holds in general, the proof is more complicated. We now give an overview of our proof. Let $H(t)$ be the long-time solution of the Hermitian Yang-Mills flow (1.6), it is well known that

$$|F_{H(t)}|_{H(t)}^2 = |F_A|_{H_0}^2. \quad (1.8)$$

So, we need only to estimate the curvature tensor $F_{H(t)}$ of Chern connection $D_{H(t)}$ with respect to the evolved Hermitian metric $H(t)$. By the assumption of theorem 1.1, there exists an exact sequence

$$0 \rightarrow \mathcal{S} \rightarrow \mathcal{E} \rightarrow \mathcal{Q} \rightarrow 0 \quad (1.9)$$

such that \mathcal{S} and \mathcal{Q} are torsion-free ω -stable sheaves, and

$$\mu_{\omega}(\mathcal{S}) > \mu_{\omega}(\mathcal{E}) > \mu_{\omega}(\mathcal{Q}). \quad (1.10)$$

On $M \setminus \Sigma_{alg}$, \mathcal{S} and \mathcal{Q} can be seen as holomorphic vector bundles, we denote $H_S(t)$ and $H_Q(t)$ the Hermitian metrics on the subbundle \mathcal{S} and the quotient bundle \mathcal{Q} induced by the evolved metric $H(t)$ on E , and $\gamma(t)$ the second fundamental form. We derive the evolution equations for $H_S(t)$, $H_Q(t)$ and $\gamma(t)$ (see (2.16), (2.17) and (2.19)). In [41], Simpson generalizes the Hermitian-Yang-Mills flow to Higgs bundle case. Under the assumption of stability, by using a result of Uhlenbeck and Yau ([45]), Simpson obtains a uniform C^0 -estimate on $H(t)$, this implies uniform higher order estimates including the uniform curvature estimate. When \mathcal{E} is unstable, the C^0 -norm of the evolved metrics $H(t)$ might be unbounded. In our case, we can obtain a uniform local C^0 -bound on the rescaled metrics $\tilde{H}_S(t) = e^{2(\lambda_S - \lambda_E)t} H_S(t)$ and $\tilde{H}_Q(t) = e^{2(\lambda_Q - \lambda_E)t} H_Q(t)$ away from the algebraic singular set Σ_{alg} . This uniform local C^0 -estimates is a key point in the proof of our theorem, where we will use the stabilities of \mathcal{S} and \mathcal{Q} and the property that $\mu_{\omega}(\mathcal{S}) > \mu_{\omega}(\mathcal{Q})$, see section 3 for details. By using the above local C^0 -estimates, we prove that the norms of the second fundamental forms $|\gamma(t)|_{H(t)}$, $|T_S(t)|_{H_S(t)}$ and $|T_Q(t)|_{H_Q(t)}$ are uniformly bounded away from Σ_{alg} . By choosing suitable test functions and using the maximum principle, we obtain a uniform local estimate of $|F_{H(t)}|_{H(t)}$ on $M \setminus \Sigma_{alg}$. In [10], under some technical assumptions on the growth of norms of the second fundamental forms associated to the HNS filtration, Collins and Jacob obtain the above uniform curvature estimate.

This paper is organized as follows. In Section 2, we recall some basic estimates for the Hermitian-Yang-Mills flow, and derive the evolution equations for the induced metrics $H_S(t)$ and $H_Q(t)$ and the second fundamental forms $\gamma(t)$. In section 3, we recall the resolution of the HNS filtration of holomorphic vector bundle, prove the related Donaldson's functionals are uniformly bounded from below, and obtain a uniform local C^0 -bound on the rescaled metrics. In section 4, we obtain a uniform local estimate for the norms of the second fundamental forms and a uniform local C^1 -estimate for the rescaled metrics. Then we complete the proof of Theorem 1.1 in section 5.

2. EVOLUTION OF THE SECOND FUNDAMENTAL FORM

Let (M, ω) be a Kähler manifold which may be noncompact, and $(E, \bar{\partial}_E)$ be a holomorphic vector bundle on M . We suppose that there is an exact sequence of holomorphic vector bundles:

$$0 \rightarrow S \rightarrow E \rightarrow Q \rightarrow 0. \quad (2.1)$$

We denote $\bar{\partial}_S$ and $\bar{\partial}_Q$ the holomorphic structures on the holomorphic subbundle S and the quotient bundle Q induced by $\bar{\partial}_E$, $\bar{\partial}_{S \oplus Q}$ the induced holomorphic structure on the direct sum bundle $S \oplus Q$, i.e.

$$\bar{\partial}_{S \oplus Q} = \begin{pmatrix} \bar{\partial}_S & 0 \\ 0 & \bar{\partial}_Q \end{pmatrix}. \quad (2.2)$$

For every Hermitian metric H on E , we have the following bundle isomorphism

$$f_H : S \oplus Q \rightarrow E, \quad (X, [Y]) \mapsto i(X) + (\text{Id}_E - \pi_H)(Y), \quad (2.3)$$

where $X \in S$, $Y \in E$, $i : S \hookrightarrow E$ is the inclusion and $\pi_H : E \rightarrow E$ is the orthogonal projection into S with respect to the metric H . In the following, we denote H_S and H_Q the Hermitian metrics on S and Q induced by the metric H on E . By the definition, the pulling back metric is

$$f_H^*(H) = \begin{pmatrix} H_S & 0 \\ 0 & H_Q \end{pmatrix}. \quad (2.4)$$

Since $\bar{\partial}_E^2 = 0$, the pull back holomorphic structure

$$f_H^*(\bar{\partial}_E) = f_H^{-1} \circ \bar{\partial}_E \circ f_H \quad (2.5)$$

is also a holomorphic structure on $S \oplus Q$. Recall that S is a holomorphic subbundle of $(E, \bar{\partial}_E)$, for any $e \in S$ and $[Y] \in Q$, we have

$$f_H^*(\bar{\partial}_E)(e, [Y]) = (\bar{\partial}_S e + \gamma([Y]), \bar{\partial}_Q [Y]), \quad (2.6)$$

where $\gamma([Y]) = -(\bar{\partial}_E \pi_H)(Y)$. So we have the following expression

$$f_H^*(\bar{\partial}_E) - \bar{\partial}_{S \oplus Q} = \begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix}, \quad (2.7)$$

where $\gamma(t) \in \Omega^{0,1}(\text{Hom}(Q, S))$ will be called the second fundamental form. Furthermore,

$$\bar{\partial}_{S \otimes Q^*} \gamma = \bar{\partial}_S \circ \gamma + \gamma \circ \bar{\partial}_Q = 0. \quad (2.8)$$

Let $D_{(\bar{\partial}_E, H)}$ be the Chern connection determined by the holomorphic structure $\bar{\partial}_E$ and the metric H , it is easy to see that the pulling back connection $f_H^*(D_{(\bar{\partial}_E, H)})$ is just the Chern connection on the holomorphic bundle $(S \oplus Q, f_H^*(\bar{\partial}_E))$ with respect to the metric $f_H^*(H)$, so

$$f_H^*(\partial_H) = \begin{pmatrix} \partial_{H_S} & 0 \\ -\gamma(t)^{*H} & \partial_{H_Q} \end{pmatrix}, \quad (2.9)$$

where $\partial_H = D_{(\bar{\partial}_E, H)}^{1,0}$, $\partial_{H_S} = D_{(\bar{\partial}_S, H_S)}^{1,0}$, $\partial_{H_Q} = D_{(\bar{\partial}_Q, H_Q)}^{1,0}$ and $\gamma(t)^{*H} \in \Omega^{1,0}(S^* \otimes Q)$ is the adjoint of γ with respect to metrics H_S and H_Q . We also have the pulling back curvature form, i.e. the following Gauss-Codazzi equation

$$f_H^*(F_H) = \begin{pmatrix} F_{H_S} - \gamma \wedge \gamma^{*H} & \partial_{H_S} \gamma + \gamma \partial_{H_Q} \\ -\bar{\partial}_Q \gamma^{*H} - \gamma^{*H} \bar{\partial}_S & F_{H_Q} - \gamma^{*H} \wedge \gamma \end{pmatrix}, \quad (2.10)$$

where F_{H_S} and F_{H_Q} are the curvature forms of the Chern connections $D_{(\bar{\partial}_S, H_S)}$ and $D_{(\bar{\partial}_Q, H_Q)}$.

Let $H(t)$ be the solution of the Hermitian-Yang-Mills flow (1.6) on the holomorphic bundle $(E, \bar{\partial}_E)$ with initial metric H_0 . Now, we will split the Hermitian-Yang-Mills flow (1.6) to the subbundle S and the quotient bundle Q . Since $f_{H(t)}|_S = i$, we have

$$\frac{\partial f_{H(t)}}{\partial t}(e) = \frac{\partial}{\partial t}(f_{H(t)}(e)) = 0, \quad (2.11)$$

$$\frac{\partial f_{H(t)}}{\partial t}([Y]) = \frac{\partial}{\partial t}((\text{Id} - \pi_{H(t)})Y) \in S, \quad (2.12)$$

for every $e \in S$ and $Y \in E$. So, we have the following expression

$$f_{H(t)}^{-1} \frac{\partial f_{H(t)}}{\partial t}(e_1, \dots, e_s, \xi_1, \dots, \xi_q) = (e_1, \dots, e_s, \xi_1, \dots, \xi_q) \begin{pmatrix} 0 & \chi(t) \\ 0 & 0 \end{pmatrix}, \quad (2.13)$$

where $\{e_j\}_{j=1}^s$ ($\{\xi_\alpha\}_{\alpha=1}^q$) is a local basis of the bundle S (resp. Q), and $\chi(t) \in \Gamma(S \otimes Q^*)$. For simplicity, we denote $\bar{H}(t) = f_{H(t)}^*(H(t))$. For every $X, Y \in S \oplus Q$, we have

$$\begin{aligned} \langle \bar{H}^{-1} \frac{\partial \bar{H}}{\partial t}(X), Y \rangle_{\bar{H}} &= \frac{\partial}{\partial t} \bar{H}(X, Y) = \frac{\partial}{\partial t} \langle X, Y \rangle_{\bar{H}} \\ &= \frac{\partial}{\partial t} \langle f_H(X), f_H(Y) \rangle_H \\ &= \frac{\partial H}{\partial t} (f_H(X), f_H(Y)) + \langle \frac{\partial f_H}{\partial t}(X), f_H(Y) \rangle_H \\ &\quad + \langle f_H(X), \frac{\partial f_H}{\partial t}(Y) \rangle_H \\ &= \langle H^{-1} \frac{\partial H}{\partial t}(f_H(X)), f_H(Y) \rangle_H + \langle f_H^{-1} \frac{\partial f_H}{\partial t}(X), Y \rangle_{\bar{H}} \\ &\quad + \langle X, f_H^{-1} \frac{\partial f_H}{\partial t}(Y) \rangle_{\bar{H}} \\ &= \langle f_H^*(H^{-1} \frac{\partial H}{\partial t})(X), Y \rangle_{\bar{H}} + \langle f_H^{-1} \frac{\partial f_H}{\partial t}(X), Y \rangle_{\bar{H}} \\ &\quad + \langle X, f_H^{-1} \frac{\partial f_H}{\partial t}(Y) \rangle_{\bar{H}}, \end{aligned} \quad (2.14)$$

and then

$$f_H^*(H^{-1} \frac{\partial H}{\partial t}) = \begin{pmatrix} H_S^{-1} \frac{\partial H_S}{\partial t} & 0 \\ 0 & H_Q^{-1} \frac{\partial H_Q}{\partial t} \end{pmatrix} - f_H^{-1} \frac{\partial f_H}{\partial t} - (f_H^{-1} \frac{\partial f_H}{\partial t})^{*H}. \quad (2.15)$$

By (1.6) and the Gauss-Codazzi equation (2.10), we have

$$H_S^{-1} \frac{\partial H_S}{\partial t} = -2(\sqrt{-1} \Lambda_\omega (F_{H_S} - \gamma \wedge \gamma^*) - \lambda \text{Id}), \quad (2.16)$$

$$H_Q^{-1} \frac{\partial H_Q}{\partial t} = -2(\sqrt{-1} \Lambda_\omega (F_{H_Q} - \gamma^* \wedge \gamma) - \lambda \text{Id}), \quad (2.17)$$

$$f_H^{-1} \frac{\partial f_H}{\partial t} = \begin{pmatrix} 0 & 2\sqrt{-1} \Lambda_\omega (\partial_{H_S} \gamma + \gamma \partial_{H_Q}) \\ 0 & 0 \end{pmatrix}. \quad (2.18)$$

Now, we consider the evolution of the second fundamental form $\gamma(t)$, i.e. we have:

Lemma 2.1. *Let $H(t)$ be the solution of the heat flow (1.6) with initial metric H_0 and $\gamma(t)$ be the second fundamental form defined by the formula (2.7), then we have:*

$$\frac{\partial}{\partial t}\gamma = 2\bar{\partial}_{S\otimes Q^*}(\sqrt{-1}\Lambda_\omega(\partial_{H_S}\gamma + \gamma\partial_{H_Q})). \quad (2.19)$$

Proof. For simplicity, we denote

$$\bar{\partial}f_H = \bar{\partial}_E \circ f_H - f_H \circ \bar{\partial}_{S\oplus Q} \quad (2.20)$$

and then

$$f_H^{-1}\bar{\partial}f_H = f_H^*(\bar{\partial}_E) - \bar{\partial}_{S\oplus Q} = \begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix}. \quad (2.21)$$

Taking the derivative of the above equation with respect to t , then

$$\begin{aligned} \begin{pmatrix} 0 & \frac{\partial\gamma}{\partial t} \\ 0 & 0 \end{pmatrix} &= \frac{\partial}{\partial t}(f_H^{-1}\bar{\partial}f_H) \\ &= -f_H^{-1}\frac{\partial f_H}{\partial t}f_H^{-1}\bar{\partial}f_H + f_H^{-1}\frac{\partial}{\partial t}(\bar{\partial}f_H) \\ &= -f_H^{-1}\frac{\partial f_H}{\partial t}f_H^{-1}\bar{\partial}f_H + f_H^{-1} \circ \bar{\partial}_E \circ f_H \circ f_H^{-1}\frac{\partial f_H}{\partial t} \\ &\quad - f_H^{-1}\frac{\partial f_H}{\partial t} \circ \bar{\partial}_{S\oplus Q} \\ &= -\begin{pmatrix} 0 & 2\sqrt{-1}\Lambda_\omega(\partial_{H_S}\gamma + \gamma\partial_{H_Q}) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix} \\ &\quad + \begin{pmatrix} \bar{\partial}_S & \gamma \\ 0 & \bar{\partial}_Q \end{pmatrix} \begin{pmatrix} 0 & 2\sqrt{-1}\Lambda_\omega(\partial_{H_S}\gamma + \gamma\partial_{H_Q}) \\ 0 & 0 \end{pmatrix} \\ &\quad - \begin{pmatrix} 0 & 2\sqrt{-1}\Lambda_\omega(\partial_{H_S}\gamma + \gamma\partial_{H_Q}) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{\partial}_S & 0 \\ 0 & \bar{\partial}_Q \end{pmatrix} \\ &= \begin{pmatrix} 0 & 2\bar{\partial}_{S\otimes Q^*}(\sqrt{-1}\Lambda_\omega(\partial_{H_S}\gamma + \gamma\partial_{H_Q})) \\ 0 & 0 \end{pmatrix}, \end{aligned} \quad (2.22)$$

i.e. we have the formula (2.19). □

Lemma 2.2. *Let $f_{H(t)}$ be the bundle isomorphism defined in (2.3), then we have*

$$f_{H_0}^{-1}f_{H(t)} = \begin{pmatrix} \text{Id}_S & G(t) \\ 0 & \text{Id}_Q \end{pmatrix}, \quad (2.23)$$

where $G(t) \in \Gamma(S \otimes Q^*)$ and $G(0) = 0$. Furthermore,

$$\frac{\partial}{\partial t}G = 2\sqrt{-1}\Lambda_\omega(\partial_{H_S}\gamma + \gamma\partial_{H_Q}), \quad (2.24)$$

and

$$\bar{\partial}_{S\otimes Q^*}G = \gamma - \gamma_0. \quad (2.25)$$

Proof. By the definition, we have

$$f_{H_0}^{-1} f_H(e) = f_{H_0}^{-1}(e) = e, \quad (2.26)$$

and

$$f_{H_0}^{-1} f_H([Y]) = f_{H_0}^{-1}((\text{Id} - \pi_H)Y) = [Y] + (\pi_{H_0} - \pi_H)Y, \quad (2.27)$$

for every $e \in S$ and $Y \in E$. So we have the expression (2.23).

Taking the derivative of the equation (2.23) with respect to t and using the formula (2.18), we get

$$\begin{aligned} \frac{\partial}{\partial t}(f_0^{-1} f_H) &= f_0^{-1} f_H \circ f_H^{-1} \frac{\partial f_H}{\partial t} \\ &= \begin{pmatrix} \text{Id}_S & G \\ 0 & \text{Id}_Q \end{pmatrix} \begin{pmatrix} 0 & 2\sqrt{-1}\Lambda_\omega(\partial_{H_S}\gamma + \gamma\partial_{H_Q}) \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 2\sqrt{-1}\Lambda_\omega(\partial_{H_S}\gamma + \gamma\partial_{H_Q}) \\ 0 & 0 \end{pmatrix}, \end{aligned} \quad (2.28)$$

i.e. we obtain the formula (2.24).

On the other hand, we have

$$\begin{aligned} \begin{pmatrix} 0 & \bar{\partial}_{S \otimes Q^*} G \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} 0 & \bar{\partial}_S \circ G - G \circ \bar{\partial}_Q \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \bar{\partial}_S & 0 \\ 0 & \bar{\partial}_Q \end{pmatrix} \begin{pmatrix} \text{Id}_S & G \\ 0 & \text{Id}_Q \end{pmatrix} - \begin{pmatrix} \text{Id}_S & G \\ 0 & \text{Id}_Q \end{pmatrix} \begin{pmatrix} \bar{\partial}_S & 0 \\ 0 & \bar{\partial}_Q \end{pmatrix} \\ &= \bar{\partial}_{S \oplus Q}(f_0^{-1} f_H) \\ &= \bar{\partial}_{S \oplus Q} \circ f_0^{-1} f_H - f_0^{-1} f_H \circ \bar{\partial}_{S \oplus Q} \\ &= \bar{\partial}_{S \oplus Q} \circ f_0^{-1} f_H - f_0^{-1} \circ \bar{\partial}_E \circ f_H \\ &\quad + f_0^{-1} \circ \bar{\partial}_E \circ f_H - f_0^{-1} f_H \circ \bar{\partial}_{S \oplus Q} \\ &= \bar{\partial}_{S \oplus Q} \circ f_0^{-1} f_H - f_0^*(\bar{\partial}_E) \circ f_0^{-1} f_H \\ &\quad + f_0^{-1} f_H \circ f_H^*(\bar{\partial}_E) - f_0^{-1} f_H \circ \bar{\partial}_{S \oplus Q} \\ &= \{\bar{\partial}_{S \oplus Q} - f_0^*(\bar{\partial}_E)\} \circ f_0^{-1} f_H + f_0^{-1} f_H \circ \{f_H^*(\bar{\partial}_E) - \bar{\partial}_{S \oplus Q}\} \\ &= - \begin{pmatrix} 0 & \gamma_0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \text{Id}_S & G \\ 0 & \text{Id}_Q \end{pmatrix} + \begin{pmatrix} \text{Id}_S & G \\ 0 & \text{Id}_Q \end{pmatrix} \begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \gamma - \gamma_0 \\ 0 & 0 \end{pmatrix}, \end{aligned} \quad (2.29)$$

so we obtain the formula (2.25). □

In the following, we consider the parabolic inequalities for $|\gamma(t)|_{H(t)}^2 = -\sqrt{-1}\Lambda_\omega \text{tr}(\gamma \wedge \gamma^{*H(t)})$ and $|G(t)|_{H(t)}^2 = \text{tr}(G \circ G^{*H(t)})$, which will be needed in the next section. By direct calculations,

we get

$$\begin{aligned}
\Delta|\gamma(t)|_{H(t)}^2 &= 2g^{k\bar{l}}\partial_k\bar{\partial}_l|\gamma(t)|_{H(t)}^2 \\
&= 2g^{k\bar{l}}\{\langle\nabla_{\partial_k}^{H(t)}\nabla_{\bar{\partial}_l}^{H(t)}\gamma, \gamma\rangle_{H(t)} + \langle\gamma, \nabla_{\bar{\partial}_k}^{H(t)}\nabla_{\partial_l}^{H(t)}\gamma\rangle_{H(t)} \\
&\quad + \langle\nabla_{\partial_k}^{H(t)}\gamma, \nabla_{\bar{\partial}_l}^{H(t)}\gamma\rangle_{H(t)} + \langle\nabla_{\bar{\partial}_k}^{H(t)}\gamma, \nabla_{\partial_l}^{H(t)}\gamma\rangle_{H(t)}\} \\
&= 2Re\{g^{k\bar{l}}\langle\nabla_{\partial_k}^{H(t)}\nabla_{\bar{\partial}_l}^{H(t)}\gamma + \nabla_{\bar{\partial}_k}^{H(t)}\nabla_{\partial_l}^{H(t)}\gamma, \gamma\rangle_{H(t)}\} \\
&\quad + 2|\nabla^{H(t)}\gamma|_{H(t)}^2,
\end{aligned} \tag{2.30}$$

$$\begin{aligned}
\nabla_{\bar{\partial}_l}^{H(t)}\gamma &= \nabla_{\bar{\partial}_l}^{H(t)}(\gamma_{\bar{j}}d\bar{z}^j) \\
&= (D_{\bar{\partial}_l}^{H(t)}\gamma_{\bar{j}})d\bar{z}^j + \gamma_{\bar{j}}\nabla_{\bar{\partial}_l}d\bar{z}^j,
\end{aligned} \tag{2.31}$$

$$\begin{aligned}
\nabla_{\partial_k}^{H(t)}\nabla_{\bar{\partial}_l}^{H(t)}\gamma &= (D_{\partial_k}^{H(t)}D_{\bar{\partial}_l}^{H(t)}\gamma_{\bar{j}})d\bar{z}^j + \gamma_{\bar{j}}(\nabla_{\partial_k}\nabla_{\bar{\partial}_l}d\bar{z}^j) \\
&\quad + (D_{\partial_k}^{H(t)}\gamma_{\bar{j}})\nabla_{\bar{\partial}_l}d\bar{z}^j,
\end{aligned} \tag{2.32}$$

and

$$\begin{aligned}
D_{\partial_k}^{H(t)}D_{\bar{\partial}_l}^{H(t)}\gamma_{\bar{j}} - D_{\bar{\partial}_l}^{H(t)}D_{\partial_k}^{H(t)}\gamma_{\bar{j}} \\
= F_{H_S}(\partial_k, \bar{\partial}_l)\gamma_{\bar{j}} - \gamma_{\bar{j}}F_{H_Q}(\partial_k, \bar{\partial}_l).
\end{aligned} \tag{2.33}$$

Since $\bar{\partial}_{S\otimes Q^*}\gamma = 0$, we have $D_{\bar{\partial}_l}^{H(t)}\gamma_{\bar{j}} = D_{\bar{\partial}_j}^{H(t)}\gamma_{\bar{l}}$, and

$$\begin{aligned}
&g^{k\bar{l}}(D_{\partial_k}^{H(t)}D_{\bar{\partial}_l}^{H(t)}\gamma_{\bar{j}} + D_{\bar{\partial}_l}^{H(t)}D_{\partial_k}^{H(t)}\gamma_{\bar{j}})d\bar{z}^j \\
&= 2g^{k\bar{l}}(D_{\partial_k}^{H(t)}D_{\bar{\partial}_l}^{H(t)}\gamma_{\bar{j}})d\bar{z}^j \\
&\quad - g^{k\bar{l}}(F_{H_S}(\partial_k, \bar{\partial}_l)\gamma_{\bar{j}} - \gamma_{\bar{j}}F_{H_Q}(\partial_k, \bar{\partial}_l))d\bar{z}^j \\
&= 2g^{k\bar{l}}(D_{\partial_k}^{H(t)}D_{\bar{\partial}_j}^{H(t)}\gamma_{\bar{l}})d\bar{z}^j \\
&\quad - (\sqrt{-1}\Lambda_\omega F_{H_S} \circ \gamma - \gamma \circ \sqrt{-1}\Lambda_\omega F_{H_Q}) \\
&= 2g^{k\bar{l}}(D_{\bar{\partial}_j}^{H(t)}D_{\partial_k}^{H(t)}\gamma_{\bar{l}})d\bar{z}^j \\
&\quad + 2g^{k\bar{l}}(F_{H_S}(\partial_k, \bar{\partial}_j)\gamma_{\bar{l}} - \gamma_{\bar{l}}F_{H_Q}(\partial_k, \bar{\partial}_j))d\bar{z}^j \\
&\quad - (\sqrt{-1}\Lambda_\omega F_{H_S} \circ \gamma - \gamma \circ \sqrt{-1}\Lambda_\omega F_{H_Q}).
\end{aligned} \tag{2.34}$$

On the other hand, it is clear that

$$\begin{aligned}
&\bar{\partial}_{S\otimes Q^*}(\sqrt{-1}\Lambda_\omega\partial^{H(t)}\gamma) = \bar{\partial}_{S\otimes Q^*}(g^{k\bar{l}}D_{\partial_k}^{H(t)}\gamma_{\bar{l}}) \\
&= 2g^{k\bar{l}}(D_{\bar{\partial}_j}^{H(t)}D_{\partial_k}^{H(t)}\gamma_{\bar{l}})d\bar{z}^j + (D_{\partial_k}^{H(t)}\gamma_{\bar{l}})\frac{\partial g^{k\bar{l}}}{\partial\bar{z}^j}d\bar{z}^j \\
&= 2g^{k\bar{l}}(D_{\bar{\partial}_j}^{H(t)}D_{\partial_k}^{H(t)}\gamma_{\bar{l}})d\bar{z}^j + (D_{\partial_k}^{H(t)}\gamma_{\bar{j}})g^{k\bar{l}}\nabla_{\bar{\partial}_l}d\bar{z}^j.
\end{aligned} \tag{2.35}$$

The above equalities yield

$$\begin{aligned}
\Delta|\gamma(t)|_{H(t)}^2 &= 2|\nabla^{H(t)}\gamma|_{H(t)}^2 + 2Ric_\omega(\partial_k, \bar{\partial}_j)g^{k\bar{l}}g^{i\bar{j}}\text{tr}(\gamma_{\bar{l}}H_Q^{-1}(\overline{\gamma_{\bar{i}}})^T H_S) \\
&\quad + 4Re\{g^{k\bar{l}}\langle(F_{H_S}(\partial_k, \bar{\partial}_j)\gamma_{\bar{l}} - \gamma_{\bar{l}}F_{H_Q}(\partial_k, \bar{\partial}_j))d\bar{z}^j, \gamma\rangle_{H(t)}\} \\
&\quad - 2Re\{(\sqrt{-1}\Lambda_\omega F_{H_S} \circ \gamma - \gamma \circ \sqrt{-1}\Lambda_\omega F_{H_Q}), \gamma\rangle_{H(t)}\} \\
&\quad + 4Re\langle\bar{\partial}_{S\otimes Q^*}(\sqrt{-1}\Lambda_\omega\partial^{H(t)}\gamma), \gamma\rangle_{H(t)}.
\end{aligned} \tag{2.36}$$

Combining (2.19), (2.16) and (2.17), we have

$$\begin{aligned}
\frac{\partial}{\partial t} |\gamma(t)|_{H(t)}^2 &= \frac{\partial}{\partial t} g^{i\bar{j}} \text{tr} (\gamma_{\bar{j}} H_Q^{-1} \overline{(\gamma_i)^T} H_S) \\
&= 4Re \langle \bar{\partial}_{S \otimes Q^*} (\sqrt{-1} \Lambda_\omega \partial^{H(t)} \gamma), \gamma \rangle_{H(t)} \\
&\quad - 2Re \langle (\sqrt{-1} \Lambda_\omega (F_{H_S} - \gamma \wedge \gamma^{*H(t)}) - \lambda_E \text{Id}_S) \circ \gamma, \gamma \rangle_{H(t)} \\
&\quad + 2Re \langle \gamma \circ (\sqrt{-1} \Lambda_\omega (F_{H_Q} - \gamma^{*H(t)} \wedge \gamma) - \lambda_E \text{Id}_Q), \gamma \rangle_{H(t)}.
\end{aligned} \tag{2.37}$$

Then, from (2.36) and (2.37), we see that

$$\begin{aligned}
&(\Delta - \frac{\partial}{\partial t}) |\gamma(t)|_{H(t)}^2 \\
&= 2|\nabla^{H(t)} \gamma|_{H(t)}^2 + 2Ric_\omega(\partial_k, \bar{\partial}_j) g^{k\bar{l}} g^{i\bar{j}} \text{tr} (\gamma_{\bar{l}} H_Q^{-1} \overline{(\gamma_i)^T} H_S) \\
&\quad + 4Re \{ g^{k\bar{l}} \langle (F_{H_S}(\partial_k, \bar{\partial}_j) \gamma_{\bar{l}} - \gamma_{\bar{l}} F_{H_Q}(\partial_k, \bar{\partial}_j)) d\bar{z}^j, \gamma \rangle_{H(t)} \} \\
&\quad + 2 \langle (-\sqrt{-1} \Lambda_\omega \gamma \wedge \gamma^{*H(t)}) \circ \gamma + \gamma \circ (\sqrt{-1} \Lambda_\omega \gamma^{*H(t)} \wedge \gamma), \gamma \rangle_{H(t)}.
\end{aligned} \tag{2.38}$$

By direct calculations, we obtain

$$\begin{aligned}
\Delta |G(t)|_{H(t)}^2 &= 2g^{k\bar{l}} \partial_k \bar{\partial}_l |G(t)|_{H(t)}^2 \\
&= 2g^{k\bar{l}} \{ \langle \nabla_{\partial_k}^{H(t)} \nabla_{\bar{\partial}_l}^{H(t)} G, G \rangle_{H(t)} + \langle G, \nabla_{\bar{\partial}_k}^{H(t)} \nabla_{\partial_l}^{H(t)} G \rangle_{H(t)} \\
&\quad + \langle \nabla_{\partial_k}^{H(t)} G, \nabla_{\bar{\partial}_l}^{H(t)} G \rangle_{H(t)} + \langle \nabla_{\bar{\partial}_l}^{H(t)} G, \nabla_{\partial_k}^{H(t)} G \rangle_{H(t)} \} \\
&= 2Re \{ g^{k\bar{l}} \langle \nabla_{\partial_k}^{H(t)} \nabla_{\bar{\partial}_l}^{H(t)} G + \nabla_{\bar{\partial}_l}^{H(t)} \nabla_{\partial_k}^{H(t)} G, G \rangle_{H(t)} \} \\
&\quad + 2|D_{H(t)} G|_{H(t)}^2 \\
&= 4Re \{ g^{k\bar{l}} \langle \nabla_{\partial_k}^{H(t)} \nabla_{\bar{\partial}_l}^{H(t)} G, G \rangle_{H(t)} \} + 2|D_{H(t)} G|_{H(t)}^2 \\
&\quad - 2Re \{ \langle (\sqrt{-1} \Lambda_\omega F_{H_S} \circ G - G \circ \sqrt{-1} \Lambda_\omega F_{H_Q}), G \rangle_{H(t)} \},
\end{aligned} \tag{2.39}$$

and

$$\begin{aligned}
\frac{\partial}{\partial t} |G(t)|_{H(t)}^2 &= 2Re \{ \langle \frac{\partial}{\partial t} G, G \rangle_{H(t)} \} \\
&\quad + Re \langle (H_S^{-1} \frac{\partial H_S}{\partial t}) \circ G - G \circ (H_Q^{-1} \frac{\partial H_Q}{\partial t}), G \rangle_{H(t)}.
\end{aligned} \tag{2.40}$$

Then we see (2.24), (2.16), (2.17), (2.39) and (2.40) imply

$$\begin{aligned}
&(\Delta - \frac{\partial}{\partial t}) |G(t)|_{H(t)}^2 \\
&= 4Re \{ g^{k\bar{l}} \langle \nabla_{\partial_k}^{H(t)} (\nabla_{\bar{\partial}_l}^{H(t)} G - \gamma(\bar{\partial}_l)), G \rangle_{H(t)} \} \\
&\quad + 2|D_{H(t)} G|_{H(t)}^2 \\
&\quad + 2 \langle (-\sqrt{-1} \Lambda_\omega \gamma \wedge \gamma^{*H(t)}) \circ G + G \circ (\sqrt{-1} \Lambda_\omega \gamma^{*H(t)} \wedge \gamma), G \rangle_{H(t)},
\end{aligned} \tag{2.41}$$

Using (2.25), we have

$$\begin{aligned}
&(\Delta - \frac{\partial}{\partial t}) |G(t)|_{H(t)}^2 \\
&= -4Re \{ g^{k\bar{l}} \langle \nabla_{\partial_k}^{H(t)} \gamma_0(\bar{\partial}_l), G \rangle_{H(t)} \} + 2|D_{H(t)} G|_{H(t)}^2 \\
&\quad + 2 \langle (-\sqrt{-1} \Lambda_\omega \gamma \wedge \gamma^{*H(t)}) \circ G + G \circ (\sqrt{-1} \Lambda_\omega \gamma^{*H(t)} \wedge \gamma), G \rangle_{H(t)}.
\end{aligned} \tag{2.42}$$

Considering the second fundamental form $\gamma_0 \in \Omega^{0,1}(S \otimes Q^*)$, since $\bar{\partial}_{S \otimes Q^*} \gamma_0 = 0$, for any point $P \in M$, we have a domain U_P and a local section $G_0 \in \Gamma(U_P; S \otimes Q^*)$ such that $\gamma_0 = \bar{\partial}_{S \otimes Q^*} G_0$. Locally, it holds that

$$\bar{\partial}_{S \otimes Q^*}(G + G_0) = \gamma. \quad (2.43)$$

Replacing G by $G + G_0$ in (2.41), we see

$$\begin{aligned} & (\Delta - \frac{\partial}{\partial t})|G(t) + G_0|_{H(t)}^2 \\ &= 4\text{Re}\{g^{k\bar{l}}\langle \nabla_{\partial_k}^{H(t)}(\nabla_{\bar{\partial}_l}^{H(t)}(G + G_0) - \gamma(\bar{\partial}_l)), G + G_0 \rangle_{H(t)}\} \\ & \quad + 2|D_{H(t)}(G + G_0)|_{H(t)}^2 \\ & \quad + 2\langle (-\sqrt{-1}\Lambda_\omega \gamma \wedge \gamma^{*H(t)}) \circ (G + G_0), (G + G_0) \rangle_{H(t)} \\ & \quad + 2\langle (G + G_0) \circ (\sqrt{-1}\Lambda_\omega \gamma^{*H(t)} \wedge \gamma), (G + G_0) \rangle_{H(t)} \\ &= 2|D_{H(t)}(G + G_0)|_{H(t)}^2 \\ & \quad + 2\langle (-\sqrt{-1}\Lambda_\omega \gamma \wedge \gamma^{*H(t)}) \circ (G + G_0), (G + G_0) \rangle_{H(t)} \\ & \quad + 2\langle (G + G_0) \circ (\sqrt{-1}\Lambda_\omega \gamma^{*H(t)} \wedge \gamma), (G + G_0) \rangle_{H(t)}. \end{aligned} \quad (2.44)$$

3. C^0 -ESTIMATE FOR THE RESCALED METRICS

Let (M, ω) be a compact Kähler manifold, and $\mathcal{E} = (E, \bar{\partial}_E)$ be a nonsemistable holomorphic vector bundle on M . We suppose that the length of the HNS-filtration of \mathcal{E} is one, i.e. there exists an exact sequence

$$0 \rightarrow \mathcal{S} \rightarrow \mathcal{E} \rightarrow \mathcal{Q} \rightarrow 0, \quad (3.1)$$

such that \mathcal{S} is an ω -stable subsheaf, and \mathcal{Q} is an ω -stable torsion-free coherent sheaf. We denote the singular set of \mathcal{Q} by Σ_{alg} . Since \mathcal{E} is nonsemistable, we have

$$\mu_\omega(\mathcal{S}) > \mu_\omega(\mathcal{E}) > \mu_\omega(\mathcal{Q}). \quad (3.2)$$

Let $H(t)$ be the solution of the Hermitian-Yang-Mills flow (1.6) on the holomorphic bundle $(E, \bar{\partial}_E)$ with initial metric H_0 . As in the above section, we denote $H_{\mathcal{S}}(t)$ and $H_{\mathcal{Q}}(t)$ the Hermitian metrics on $\mathcal{S}|_{M \setminus \Sigma_{alg}}$ and $\mathcal{Q}|_{M \setminus \Sigma_{alg}}$ induced by the metric $H(t)$ on E . In this section, we will derive a uniform local C^0 -estimate for the rescaled metric $H_{\mathcal{S}}(t) = e^{2(\lambda_{\mathcal{S}} - \lambda_{\mathcal{E}})t} H_{\mathcal{S}}(t)$ and $H_{\mathcal{Q}}(t) = e^{2(\lambda_{\mathcal{Q}} - \lambda_{\mathcal{E}})t} H_{\mathcal{Q}}(t)$ outside Σ_{alg} , where $\lambda_{\mathcal{S}} = \frac{\mu_\omega(\mathcal{S})}{\text{Vol}(M, \omega)}$ and $\lambda_{\mathcal{Q}} = \frac{\mu_\omega(\mathcal{Q})}{\text{Vol}(M, \omega)}$.

By Hironaka's flattening theorem ([19], [20]), we have a resolution of the HNS-filtration ([39]), by successively blowing up $\pi_j : M_j \rightarrow M_{j-1}$ with smooth center $Y_{i-1} \subset M_{i-1}$ finite times, where $j = 1, \dots, k$, $M_0 = M$, $\tilde{M} = M_k$, there is an exact sequence on \tilde{M}

$$0 \rightarrow \mathcal{S} \rightarrow \tilde{\mathcal{E}} \rightarrow \mathcal{Q} \rightarrow 0 \quad (3.3)$$

such that: (1) \mathcal{S}, \mathcal{Q} are locally free; (2) the composition $\pi = \pi_1 \circ \dots \circ \pi_k : \tilde{M} \rightarrow M$ is biholomorphic outside Σ_{alg} ; (3) $\tilde{\mathcal{E}} = \pi^* \mathcal{E}$; (4) \mathcal{S}, \mathcal{Q} are isomorphic to the sheaves \mathcal{S}, \mathcal{Q} respectively outside $\pi^{-1}\Sigma_{alg}$, $\pi_* \mathcal{S} = \mathcal{S}$ and $\mathcal{Q}^{**} = (\pi_* \mathcal{Q})^{**}$.

It is well known that \tilde{M} is also Kähler ([16]). As in [7], we fix arbitrary Kähler metrics η_i on M_i and set

$$\omega_{1,\epsilon} = \pi_1^* \omega + \epsilon_1 \eta_1, \quad \omega_{i,\epsilon} = \pi_i^* \omega_{i-1,\epsilon} + \epsilon_i \eta_i \quad (3.4)$$

for all $1 \leq i \leq k$ and $0 < \epsilon_i \leq 1$, where $\epsilon = (\epsilon_1, \dots, \epsilon_k)$. For the sake of simplicity, we only consider the case that $k = 1$, the general case following by induction.

In the following, we always assume that $\pi : \tilde{M} \rightarrow M$ is a single blow-up with smooth centre, fix a Kähler metric η on \tilde{M} and set $\omega_\epsilon = \pi^* \omega + \epsilon \eta$ for $0 < \epsilon \leq 1$. In fact there exists a

holomorphic line bundle L over \tilde{M} with respect to the exceptional divisor $D \subset \pi^{-1}(\Sigma_{alg})$ such that the $(1, 1)$ -form $\pi^*\omega + \delta\sqrt{-1}F_{H_L}$ is positive for some δ small enough (for the proof see for example [16, 46]), where $\sqrt{-1}F_{H_L}$ is the Chern form with respect to some Hermitian metric on L . In the following, we can set the Kähler metric η by

$$\eta = \pi^*\omega + \delta \cdot \sqrt{-1}F_{H_L}. \quad (3.5)$$

Bando and Siu (Lemma 3 in [7]) derived a uniform Sobolev inequality for $(\tilde{M}, \omega_\epsilon)$, i.e. there exists a uniform constant C_S such that

$$\left(\int_{\tilde{M}} |\rho|^{\frac{2n}{2n-1}} \frac{\omega_\epsilon^n}{n!} \right)^{\frac{2n-1}{2n}} \leq C_S \left(\int_{\tilde{M}} |d\rho|_{\omega_\epsilon} + |\rho| \frac{\omega_\epsilon^n}{n!} \right) \quad (3.6)$$

for all $\rho \in C^1(\tilde{M})$ and all $0 < \epsilon \leq 1$. Using Li's result (Proposition 3 in [32]), we obtain a uniform lower bound on the first eigenvalue of Δ_ϵ , i.e. there exists a uniform constant C_P such that

$$C_P \cdot \inf_{a \in \mathbb{R}} \left(\int_{\tilde{M}} |\rho - a|^2 \frac{\omega_\epsilon^n}{n!} \right) \leq \int_{\tilde{M}} |d\rho|_{\omega_\epsilon}^2 \frac{\omega_\epsilon^n}{n!} \quad (3.7)$$

for all $\rho \in W^{1,2}(\tilde{M})$ and all $0 < \epsilon \leq 1$. Combining Cheng and Li's estimate ([9]) with Grigor'yan's result (Theorem 1.1 in [17]), we have the following uniform upper bounds of the heat kernels and uniform lower bounds of the Green functions.

Proposition 3.1. (Proposition 2 in [7]) *Let \mathcal{K}_ϵ be the heat kernel with respect to the metric ω_ϵ , then for any $\tau > 0$, there exists a constant $C_{\mathcal{K}}(\tau)$ independent of ϵ , such that*

$$0 \leq \mathcal{K}_\epsilon(x, y, t) \leq C_{\mathcal{K}}(\tau) \left(t^{-n} \exp \left(-\frac{(d_{\omega_\epsilon}(x, y))^2}{(4 + \tau)t} \right) + 1 \right) \quad (3.8)$$

for every $x, y \in \tilde{M}$ and $0 < t < +\infty$, where $d_{\omega_\epsilon}(x, y)$ is the distance between x and y with respect to the metric ω_ϵ . There also exists a constant $C_{\mathcal{G}}$ such that

$$\mathcal{G}_\epsilon(x, y) \geq -C_{\mathcal{G}} \quad (3.9)$$

for every $x, y \in \tilde{M}$ and $0 < \epsilon \leq 1$, where \mathcal{G}_ϵ is the Green function with respect to the metric ω_ϵ .

Let $H_\epsilon(t)$ be the solution of the Hermitian-Yang-Mills flow (1.6) on the holomorphic bundle $\tilde{\mathcal{E}}$ over the Kähler manifold $(\tilde{M}, \omega_\epsilon)$ with the fixed initial metric π^*H_0 , i.e. it satisfies

$$\begin{cases} H_\epsilon(t)^{-1} \frac{\partial H_\epsilon(t)}{\partial t} = -2(\sqrt{-1}\Lambda_{\omega_\epsilon} F_{H_\epsilon(t)} - \lambda_{\mathcal{E}, \epsilon} \text{Id}_{\tilde{\mathcal{E}}}), \\ H_\epsilon(0) = \pi^*H_0. \end{cases} \quad (3.10)$$

By Bando and Siu's argument in [7] and the uniqueness of the Hermitian-Yang-Mills flow, we know that, by choosing a subsequence, $H_\epsilon(t)$ converges to $H(t)$ in C_{loc}^∞ -topology outside Σ_{alg} as $\epsilon \rightarrow 0$. We also denote $H_{S, \epsilon}(t)$ and $H_{Q, \epsilon}(t)$ the Hermitian metrics on bundles S and Q induced by the metric $H_\epsilon(t)$. It is easy to see that: by choosing a subsequence,

$$H_{S, \epsilon}(t) \rightarrow H_S(t), \quad H_{Q, \epsilon}(t) \rightarrow H_Q(t) \quad (3.11)$$

in C_{loc}^∞ -topology outside Σ_{alg} as $\epsilon \rightarrow 0$.

Since \mathcal{S} and \mathcal{Q} are ω -stable, it is easy to check that $(S, \bar{\partial}_S)$ and $(Q, \bar{\partial}_Q)$ are ω_ϵ -stable for sufficiently small ϵ . By Donaldson-Uhlenbeck-Yau theorem, we can suppose that $K_{S, \epsilon}$ and $K_{Q, \epsilon}$ are ω_ϵ -Hermitian-Einstein metrics on $(S, \bar{\partial}_S)$ and $(Q, \bar{\partial}_Q)$, i.e.

$$\sqrt{-1}\Lambda_{\omega_\epsilon} F_{K_{S, \epsilon}} = \lambda_{S, \epsilon} \text{Id}_S \quad (3.12)$$

and

$$\sqrt{-1}\Lambda_{\omega_\epsilon} F_{K_{Q, \epsilon}} = \lambda_{Q, \epsilon} \text{Id}_Q. \quad (3.13)$$

Here $\lambda_{S,\epsilon} = \frac{\mu_{\omega_\epsilon}(S)}{\text{Vol}(M, \omega_\epsilon)}$ and $\lambda_{Q,\epsilon} = \frac{\mu_{\omega_\epsilon}(Q)}{\text{Vol}(M, \omega_\epsilon)}$, and

$$\lambda_{S,\epsilon} \rightarrow \lambda_S, \quad \lambda_{Q,\epsilon} \rightarrow \lambda_Q \quad (3.14)$$

as $\epsilon \rightarrow 0$.

Denote $h_{S,\epsilon}(t) = K_{S,\epsilon}^{-1} H_{S,\epsilon}(t)$, $h_{Q,\epsilon}(t) = K_{Q,\epsilon}^{-1} H_{Q,\epsilon}(t)$, and set $\tilde{h}_{S,\epsilon}(t) = e^{2(\lambda_{S,\epsilon} - \lambda_{\tilde{S},\epsilon})t} h_{S,\epsilon}(t)$, $\tilde{h}_{Q,\epsilon}(t) = e^{2(\lambda_{Q,\epsilon} - \lambda_{\tilde{Q},\epsilon})t} h_{Q,\epsilon}(t)$. Using (2.16) and (2.17), we have

$$\begin{aligned} (\Delta_\epsilon - \frac{\partial}{\partial t}) \text{tr} \tilde{h}_{S,\epsilon} &= 2 \text{tr} (-\sqrt{-1} \Lambda_{\omega_\epsilon} \bar{\partial} \tilde{h}_{S,\epsilon} \tilde{h}_{S,\epsilon}^{-1} \partial_{K_{S,\epsilon}} \tilde{h}_{S,\epsilon}) - 2(\lambda_{S,\epsilon} - \lambda_{\tilde{S},\epsilon}) \text{tr} (\tilde{h}_{S,\epsilon}) \\ &\quad - 2 \text{tr} (\tilde{h}_{S,\epsilon} \sqrt{-1} \Lambda_{\omega_\epsilon} (F_{H_{S,\epsilon}} - F_{K_{S,\epsilon}})) - \text{tr} (\tilde{h}_{S,\epsilon} H_{S,\epsilon}^{-1} \frac{\partial H_{S,\epsilon}}{\partial t}) \\ &= 2 \text{tr} (-\sqrt{-1} \Lambda_{\omega_\epsilon} \bar{\partial} \tilde{h}_{S,\epsilon} \tilde{h}_{S,\epsilon}^{-1} \partial_{K_{S,\epsilon}} \tilde{h}_{S,\epsilon}) \\ &\quad - 2 \text{tr} (\tilde{h}_{S,\epsilon} (\sqrt{-1} \Lambda_{\omega_\epsilon} \gamma_\epsilon \wedge \gamma_\epsilon^*)) \\ &\geq 0, \end{aligned} \quad (3.15)$$

and

$$\begin{aligned} (\Delta_\epsilon - \frac{\partial}{\partial t}) \text{tr} \tilde{h}_{Q,\epsilon}^{-1} &= 2 \text{tr} (-\sqrt{-1} \Lambda_{\omega_\epsilon} \bar{\partial} \tilde{h}_{Q,\epsilon}^{-1} \circ \tilde{h}_{Q,\epsilon} \circ \partial_{H_{Q,\epsilon}} \tilde{h}_{Q,\epsilon}^{-1}) \\ &\quad + 2 \text{tr} (\tilde{h}_{Q,\epsilon}^{-1} (\sqrt{-1} \Lambda_{\omega_\epsilon} \gamma_\epsilon^* \wedge \gamma_\epsilon)) \\ &\geq 0, \end{aligned} \quad (3.16)$$

where we have used the nonnegativity of $\sqrt{-1} \Lambda_{\omega_\epsilon} \gamma_\epsilon^* \wedge \gamma_\epsilon$. Using the above inequalities and the uniform upper bound on the heat kernels, we can get a uniform bound (independent of ϵ) on $\text{tr} \tilde{h}_{S,\epsilon} + \text{tr} \tilde{h}_{Q,\epsilon}^{-1}$. In fact, we obtain the following lemma.

Lemma 3.2. *There exists a uniform constant $C_{0,1}$ such that*

$$\sup_{x \in \tilde{M}} (\text{tr} \tilde{h}_{S,\epsilon}(x, t) + \text{tr} \tilde{h}_{Q,\epsilon}^{-1}(x, t)) \leq C_{0,1}, \quad (3.17)$$

for all $t \in [0, +\infty)$ and $0 < \epsilon \leq 1$.

Proof. In [7], Bando and Siu conjectured that $K_{S,\epsilon}$ and $K_{Q,\epsilon}$ should converge to ω -Hermitian-Einstein metrics K_S and K_Q on sheaves \mathcal{S} and \mathcal{Q} in local C^∞ -topology outside $\pi^{-1}\Sigma_{\text{alg}}$. This fact has been proved in [30]. By taking constants on $K_{S,\epsilon}$ and $K_{Q,\epsilon}$, we can suppose that

$$\int_{\tilde{M}} \log \det(\tilde{h}_{S,\epsilon}(0)) \frac{\omega_\epsilon^n}{n!} = \int_{\tilde{M}} \log \det(\tilde{h}_{Q,\epsilon}(0)) \frac{\omega_\epsilon^n}{n!} = 0. \quad (3.18)$$

By the uniform L^1 -estimate in [30] (Lemma 5.1.), we see that there exists a uniform constant \hat{C} such that

$$\int_{\tilde{M}} \log \{ \text{tr} (\tilde{h}_{S,\epsilon}(0)) + \text{tr} (\tilde{h}_{S,\epsilon}^{-1}(0)) \} + \log \{ \text{tr} (\tilde{h}_{Q,\epsilon}(0)) + \text{tr} (\tilde{h}_{Q,\epsilon}^{-1}(0)) \} \frac{\omega_\epsilon^n}{n!} \leq \hat{C} \quad (3.19)$$

for all $0 < \epsilon \leq 1$. For any point $x \in \tilde{M} \setminus \pi^{-1}(\Sigma_{\text{alg}})$ and any vector $X \in \tilde{\mathcal{E}}_x$ satisfying $|X|_{\pi^* H_0} = 1$, it is easy to check that

$$-\sup_{x \in \tilde{M}} |F_{H_0}|_{\omega, H_0}(x) \cdot \pi^* \omega \leq \langle \sqrt{-1} F_{\pi^* H_0}(X), X \rangle_{\pi^* H_0} \leq \sup_{x \in \tilde{M}} |F_{H_0}|_{\omega, H_0}(x) \cdot \pi^* \omega, \quad (3.20)$$

and

$$-n \sup_{x \in \tilde{M}} |F_{H_0}|_{\omega, H_0}(x) \leq \Lambda_{\omega_\epsilon} \langle \sqrt{-1} F_{\pi^* H_0}(X), X \rangle_{\pi^* H_0} \leq n \sup_{x \in \tilde{M}} |F_{H_0}|_{\omega, H_0}(x). \quad (3.21)$$

Then we have

$$-n \sup_{x \in \tilde{M}} |F_{H_0}|_{\omega, H_0}(x) \text{Id}_{\tilde{\mathcal{E}}} \leq \sqrt{-1} \Lambda_{\omega_\epsilon} F_{\pi^* H_0} \leq n \sup_{x \in \tilde{M}} |F_{H_0}|_{\omega, H_0}(x) \text{Id}_{\tilde{\mathcal{E}}} \quad (3.22)$$

on the whole \tilde{M} . Using the Gauss-Codazzi equation (2.10), we have

$$\sqrt{-1} \Lambda_{\omega_\epsilon} F_{H_{S,\epsilon}(0)} \leq n \sup_{x \in \tilde{M}} |F_{H_0}|_{\omega, H_0}(x) \text{Id}_S, \quad (3.23)$$

and

$$\sqrt{-1} \Lambda_{\omega_\epsilon} F_{H_{Q,\epsilon}(0)} \geq -n \sup_{x \in \tilde{M}} |F_{H_0}|_{\omega, H_0}(x) \text{Id}_Q. \quad (3.24)$$

Direct calculations yield

$$\begin{aligned} \Delta_\epsilon \text{tr } \tilde{h}_{S,\epsilon}(0) &= 2\text{tr}(-\sqrt{-1} \Lambda_{\omega_\epsilon} \bar{\partial} \tilde{h}_{S,\epsilon}(0) \tilde{h}_{S,\epsilon}^{-1}(0) \partial_{K_{S,\epsilon}} \tilde{h}_{S,\epsilon}(0)) \\ &\quad - 2\text{tr}(\tilde{h}_{S,\epsilon}(0) \sqrt{-1} \Lambda_{\omega_\epsilon} (F_{H_{S,\epsilon}(0)} - F_{K_{S,\epsilon}})) \\ &\geq 2\text{tr}(-\sqrt{-1} \Lambda_{\omega_\epsilon} \bar{\partial} \tilde{h}_{S,\epsilon}(0) \tilde{h}_{S,\epsilon}^{-1}(0) \partial_{K_{S,\epsilon}} \tilde{h}_{S,\epsilon}(0)) \\ &\quad - 2\text{tr}(\tilde{h}_{S,\epsilon}(0)) (n \sup_{x \in \tilde{M}} |F_{H_0}|_{\omega, H_0}(x) - \lambda_{S,\epsilon}), \end{aligned} \quad (3.25)$$

$$\begin{aligned} \Delta_\epsilon \text{tr } \tilde{h}_{Q,\epsilon}^{-1}(0) &\geq 2\text{tr}(-\sqrt{-1} \Lambda_{\omega_\epsilon} \bar{\partial} \tilde{h}_{Q,\epsilon}^{-1} \circ \tilde{h}_{Q,\epsilon} \circ \partial_{H_{Q,\epsilon}} \tilde{h}_{Q,\epsilon}^{-1}) \\ &\quad - 2\text{tr}(\tilde{h}_{Q,\epsilon}^{-1}(0)) (n \sup_{x \in \tilde{M}} |F_{H_0}|_{\omega, H_0}(x) + \lambda_{Q,\epsilon}). \end{aligned} \quad (3.26)$$

Then there is a uniform constant $\hat{C}_{0,1}$ such that

$$\Delta_\epsilon \log(\text{tr } \tilde{h}_{S,\epsilon}(0) + 1) \geq -\hat{C}_{0,1}, \quad (3.27)$$

and

$$\Delta_\epsilon \log(\text{tr } \tilde{h}_{Q,\epsilon}^{-1}(0) + 1) \geq -\hat{C}_{0,1}, \quad (3.28)$$

for all $0 < \epsilon \leq 1$. The uniform lower bounds of the Green functions (3.9), inequalities (3.19), (3.27) and (3.28) imply that there is a uniform constant $\tilde{C}_{0,1}$ such that

$$\sup_{x \in \tilde{M}} (\text{tr } \tilde{h}_{S,\epsilon}(0) + \text{tr } \tilde{h}_{Q,\epsilon}^{-1}(0)) \leq \tilde{C}_{0,1}, \quad (3.29)$$

for all $0 < \epsilon \leq 1$. By (3.15), (3.16) and the maximum principle, we obtain the estimate (3.17). \square

In the following we will get uniform upper bounds on $\det(\tilde{h}_{S,\epsilon}^{-1}(t))$ and $\det(\tilde{h}_{Q,\epsilon}(t))$, which imply uniform upper bounds on $\text{tr } \tilde{h}_{S,\epsilon}^{-1}(t)$ and $\text{tr } \tilde{h}_{Q,\epsilon}(t)$ by (3.17). Then we get uniform C^0 -bounds on $\tilde{h}_{S,\epsilon}(t)$ and $\tilde{h}_{Q,\epsilon}(t)$.

Let's recall

$$(\Delta_\epsilon - \frac{\partial}{\partial t}) \text{tr}(\sqrt{-1} \Lambda_{\omega_\epsilon} F_{H_\epsilon(t)} - \lambda_{\mathcal{E},\epsilon} \text{Id}_{\mathcal{E}}) = 0. \quad (3.30)$$

For simplicity, we set $f_\epsilon(t) = \text{tr}(\sqrt{-1} \Lambda_{\omega_\epsilon} F_{H_\epsilon(t)} - \lambda_{\mathcal{E},\epsilon} \text{Id}_{\mathcal{E}})$. By (3.22) and the uniform Poincaré inequality (3.7), we know that there exists a uniform constant $C_{F,1}$ such that

$$\int_{\tilde{M}} |f_\epsilon(t)|^2 \frac{\omega_\epsilon^n}{n!} \leq C_{F,1} \cdot \exp\{-2C_p t\} \quad (3.31)$$

for all $0 < \epsilon \leq 1$ and $t \in [0, +\infty)$. From the upper bound of the heat kernels (3.8), we get

$$\max_{x \in \tilde{M}} f_\epsilon^2(x, t) \leq 2C_K(\tau) \int_{\tilde{M}} |f_\epsilon(t-1)|^2 \frac{\omega_\epsilon^n}{n!} \leq 2C_K(\tau) \cdot C_{F,1} \cdot \exp\{-2C_p t\} \quad (3.32)$$

for all $0 < \epsilon \leq 1$ and $t \in [1, +\infty)$. Using the Gauss-Codazzi equation (2.10), we have

$$\begin{aligned} & \operatorname{tr}(\sqrt{-1}\Lambda_{\omega_\epsilon} F_{H_\epsilon(t)} - \lambda_{\mathcal{E},\epsilon} \operatorname{Id}_{\mathcal{E}}) \\ &= \operatorname{tr}(\sqrt{-1}\Lambda_{\omega_\epsilon} (F_{H_{S,\epsilon}(t)} - \gamma_\epsilon \wedge \gamma_\epsilon^*)) - \lambda_{S,\epsilon} \operatorname{rank}(S) \\ & \quad + \operatorname{tr}(\sqrt{-1}\Lambda_{\omega_\epsilon} (F_{H_{Q,\epsilon}(t)} - \gamma_\epsilon^* \wedge \gamma_\epsilon) - \lambda_{Q,\epsilon} \operatorname{rank}(Q)), \end{aligned} \quad (3.33)$$

and then

$$\begin{aligned} & |\sqrt{-1}\Lambda_{\omega_\epsilon} (F_{H_{S,\epsilon}(t)} - \gamma_\epsilon \wedge \gamma_\epsilon^*) - \lambda_{\mathcal{E},\epsilon} \operatorname{Id}_S|^2 \\ & + |\sqrt{-1}\Lambda_{\omega_\epsilon} (F_{H_{Q,\epsilon}(t)} - \gamma_\epsilon^* \wedge \gamma_\epsilon) - \lambda_{\mathcal{E},\epsilon} \operatorname{Id}_Q|^2 \\ &= 2(\lambda_{S,\epsilon} - \lambda_{Q,\epsilon}) \operatorname{tr}(\sqrt{-1}\Lambda_{\omega_\epsilon} (F_{H_{S,\epsilon}(t)} - \gamma_\epsilon \wedge \gamma_\epsilon^*) - \lambda_{S,\epsilon} \operatorname{Id}_S) \\ & \quad + 2(\lambda_{Q,\epsilon} - \lambda_{\mathcal{E},\epsilon}) \operatorname{tr}(\sqrt{-1}\Lambda_{\omega_\epsilon} F_{H_\epsilon(t)} - \lambda_{\mathcal{E},\epsilon} \operatorname{Id}_{\mathcal{E}}) \\ & \quad + |\sqrt{-1}\Lambda_{\omega_\epsilon} (F_{H_{S,\epsilon}(t)} - \gamma_\epsilon \wedge \gamma_\epsilon^*) - \lambda_{S,\epsilon} \operatorname{Id}_S|^2 \\ & \quad + |\sqrt{-1}\Lambda_{\omega_\epsilon} (F_{H_{Q,\epsilon}(t)} - \gamma_\epsilon^* \wedge \gamma_\epsilon) - \lambda_{Q,\epsilon} \operatorname{Id}_Q|^2 \\ & \quad + (\lambda_{S,\epsilon} - \lambda_{\mathcal{E},\epsilon})^2 \operatorname{rank}(S) + (\lambda_{Q,\epsilon} - \lambda_{\mathcal{E},\epsilon})^2 \operatorname{rank}(Q). \end{aligned} \quad (3.34)$$

Let's recall the Donaldson's functional

$$\mathcal{M}_{\tilde{\mathcal{E}},\epsilon}^0(H_0, H) = \int_0^1 \int_{\tilde{M}} \operatorname{tr}(\sqrt{-1}\Lambda_{\omega} F_{H(s)} H^{-1}(s) \frac{\partial H(s)}{\partial s}) \frac{\omega_\epsilon^n}{n!} \quad (3.35)$$

and

$$\mathcal{M}_{\tilde{\mathcal{E}},\epsilon}(H_0, H) = \mathcal{M}_{\tilde{\mathcal{E}}}^0(H_0, H) - \lambda_{\tilde{\mathcal{E}},\epsilon} \int_{\tilde{M}} \log \det(H_0^{-1} H) \frac{\omega_\epsilon^n}{n!}, \quad (3.36)$$

where $H(s)$ is a path connecting metrics H_0 and H on $\tilde{\mathcal{E}}$. Donaldson proved that the above integral is independent of the path, and the Hermitian-Yang-Mills flow is the gradient flow of the above functional, i.e. if $H_\epsilon(t)$ is a solution of the Hermitian-Yang-Mills flow (1.6), we have

$$\frac{d}{dt} \mathcal{M}_{\tilde{\mathcal{E}},\epsilon}(H_0, H(t)) = -2 \int_{\tilde{M}} |\sqrt{-1}\Lambda_{\omega_\epsilon} F_{H_\epsilon(t)} - \lambda_{\tilde{\mathcal{E}},\epsilon} \operatorname{Id}_{\mathcal{E}}|_{H_\epsilon(t)}^2 \frac{\omega_\epsilon^n}{n!}. \quad (3.37)$$

Since $(S, \bar{\partial}_S)$ and $(Q, \bar{\partial}_Q)$ are ω_ϵ -stable bundles over \tilde{M} for a fixed ϵ , the Donaldson's functional $\mathcal{M}_{S,\epsilon}(H_{S,0}, \cdot)$ and $\mathcal{M}_{Q,\epsilon}(H_{Q,0}, \cdot)$ are bounded from below. In the following, we will prove that the Donaldson's functional $\mathcal{M}_{S,\epsilon}(H_{S,\epsilon}(0), \cdot)$ and $\mathcal{M}_{Q,\epsilon}(H_{Q,\epsilon}(0), \cdot)$ are uniformly bounded from below.

Proposition 3.3. *There exists a uniform positive constant $C_{\mathcal{M}}$ such that*

$$\mathcal{M}_{S,\epsilon}(H_{S,\epsilon}(0), K_{S,\epsilon}) \geq -C_{\mathcal{M}} \quad (3.38)$$

and

$$\mathcal{M}_{Q,\epsilon}(H_{Q,\epsilon}(0), K_{Q,\epsilon}) \geq -C_{\mathcal{M}} \quad (3.39)$$

for all $0 < \epsilon \leq 1$, where $H_{S,\epsilon}(0)$ and $H_{Q,\epsilon}(0)$ are metrics on S and Q induced by the pull back metric $\pi^* H_0$, $K_{S,\epsilon}$ and $K_{Q,\epsilon}$ are ω_ϵ -Hermitian-Einstein metrics on S and Q .

Proof. Setting $\exp\{\varrho_{S,\epsilon}\} := h_{S,\epsilon}^{-1}(0) = H_{S,\epsilon}^{-1}(0) K_{S,\epsilon}$, $\exp\{\varrho_{Q,\epsilon}\} := h_{Q,\epsilon}^{-1}(0) = H_{Q,\epsilon}^{-1}(0) K_{Q,\epsilon}$, we have the following expression of the Donaldson's functional

$$\begin{aligned} \mathcal{M}_{S,\epsilon}(H_{S,\epsilon}(0), K_{S,\epsilon}) &= \int_{\tilde{M}} \operatorname{tr} \{ \varrho_{S,\epsilon} \cdot (\sqrt{-1}\Lambda_{\omega_\epsilon} F_{H_{S,\epsilon}(0)} - \lambda_{S,\epsilon} \operatorname{Id}_S) \} \\ & \quad + \langle \Psi(\varrho_{S,\epsilon})(\bar{\partial} \varrho_{S,\epsilon}), \bar{\partial} \varrho_{S,\epsilon} \rangle_{H_{S,\epsilon}(0)} \frac{\omega_\epsilon^n}{n!}, \end{aligned} \quad (3.40)$$

where $\Psi(x, y) = (x - y)^{-2}(e^{y-x} - (y - x) - 1)$. Since the second part of the right hand side of the above equality is nonnegative, we have

$$\mathcal{M}_{S,\epsilon}(H_{S,\epsilon}(0), K_{S,\epsilon}) \geq \int_{\tilde{M}} \text{tr} \{ \varrho_{S,\epsilon} \cdot (\sqrt{-1}\Lambda_{\omega_\epsilon} F_{H_{S,\epsilon}(0)} - \lambda_{S,\epsilon} \text{Id}_S) \} \frac{\omega_\epsilon^n}{n!}, \quad (3.41)$$

and

$$\mathcal{M}_{Q,\epsilon}(H_{Q,\epsilon}(0), K_{Q,\epsilon}) \geq \int_{\tilde{M}} \text{tr} \{ \varrho_{Q,\epsilon} \cdot (\sqrt{-1}\Lambda_{\omega_\epsilon} F_{H_{Q,\epsilon}(0)} - \lambda_{Q,\epsilon} \text{Id}_S) \} \frac{\omega_\epsilon^n}{n!}. \quad (3.42)$$

By (3.29), there must exist a uniform constant $\tilde{C}_{0,2}$ such that

$$\varrho_{S,\epsilon} \geq -\tilde{C}_{0,2} \text{Id}_S, \quad \varrho_{Q,\epsilon} \leq \tilde{C}_{0,2} \text{Id}_S \quad (3.43)$$

for all $0 < \epsilon \leq 1$. (3.19), (3.23), (3.43) and (3.41) imply that there is a uniform $\tilde{C}_{0,3}$ such that

$$\mathcal{M}_{S,\epsilon}(H_{S,\epsilon}(0), K_{S,\epsilon}) \geq \int_{\tilde{M}} \text{tr} (\varrho_{S,\epsilon}) \cdot \text{tr} (\sqrt{-1}\Lambda_{\omega_\epsilon} F_{H_{S,\epsilon}(0)}) \frac{\omega_\epsilon^n}{n!} - \tilde{C}_{0,3}, \quad (3.44)$$

and

$$\mathcal{M}_{Q,\epsilon}(H_{Q,\epsilon}(0), K_{Q,\epsilon}) \geq \int_{\tilde{M}} \text{tr} (\varrho_{Q,\epsilon}) \cdot \text{tr} (\sqrt{-1}\Lambda_{\omega_\epsilon} F_{H_{Q,\epsilon}(0)}) \frac{\omega_\epsilon^n}{n!} - \tilde{C}_{0,3}, \quad (3.45)$$

for all $0 < \epsilon \leq 1$.

By the definition, we have

$$\text{tr} \varrho_{S,\epsilon} = \log \frac{\det(K_{S,\epsilon})}{\det(H_{S,\epsilon}(0))}. \quad (3.46)$$

In the following, we will show that $\epsilon \cdot \log \frac{\det(K_{S,\epsilon})}{\det(H_{S,\epsilon}(0))}$ are uniformly bounded. Let $\tilde{H}_{S,\epsilon}(t)$ be the evolved metric along the Hermitian-Yang-Mills flow (1.6) with initial metric $H_{S,\epsilon}(0)$, i.e. it satisfies

$$\begin{cases} \tilde{H}_{S,\epsilon}(t)^{-1} \frac{\partial \tilde{H}_{S,\epsilon}(t)}{\partial t} = -2(\sqrt{-1}\Lambda_{\omega_\epsilon} F_{\tilde{H}_{S,\epsilon}(t)} - \lambda_{S,\epsilon} \text{Id}_S), \\ \tilde{H}_{S,\epsilon}(0) = H_{S,\epsilon}(0). \end{cases} \quad (3.47)$$

Since $H_{S,\epsilon}(0)$ is a fixed smooth Hermitian metric on bundle S , there is a uniform constant $\tilde{C}_{0,5}$ such that

$$-\tilde{C}_{0,5}\eta \leq \sqrt{-1}\text{tr} F_{H_{S,\epsilon}(0)} \leq \tilde{C}_{0,5}\eta, \quad (3.48)$$

for all $0 < \epsilon \leq 1$. (3.48) and (3.47) imply that

$$\begin{aligned} |\epsilon \text{tr} (\sqrt{-1}\Lambda_{\omega_\epsilon} F_{H_{S,\epsilon}(0)})| &= n\epsilon \left| \frac{\sqrt{-1}\text{tr} F_{H_{S,\epsilon}(0)} \wedge \omega_\epsilon^{n-1}}{\omega_\epsilon^n} \right| \\ &\leq n \frac{\epsilon \eta \wedge \omega_\epsilon^{n-1}}{\omega_\epsilon^n} \leq n\tilde{C}_{0,5}, \end{aligned} \quad (3.49)$$

and

$$\begin{aligned} \epsilon \left| \frac{\partial}{\partial t} \log \frac{\det(\tilde{H}_{S,\epsilon}(t))}{\det(H_{S,\epsilon}(0))} \right| &= \epsilon \left| \text{tr} (\tilde{H}_{S,\epsilon}(t)^{-1} \frac{\partial \tilde{H}_{S,\epsilon}(t)}{\partial t}) \right| \\ &= |\epsilon \text{tr} (\sqrt{-1}\Lambda_{\omega_\epsilon} F_{\tilde{H}_{S,\epsilon}(t)} - \lambda_{S,\epsilon} \text{Id}_S)| \\ &\leq \sup_{x \in \tilde{M}} |\epsilon \text{tr} (\sqrt{-1}\Lambda_{\omega_\epsilon} F_{H_{S,\epsilon}(0)} - \lambda_{S,\epsilon} \text{Id}_S)| \\ &\leq n\tilde{C}_{0,5} + n\epsilon |\lambda_{S,\epsilon}|, \end{aligned} \quad (3.50)$$

for all $0 < \epsilon \leq 1$. Integrating the inequality (3.50), we have

$$\max_{x \in \tilde{M}} \epsilon \left| \log \frac{\det(\tilde{H}_{S,\epsilon}(1))}{\det(H_{S,\epsilon}(0))} \right| \leq n\tilde{C}_{0,5} + n\epsilon |\lambda_{S,\epsilon}|, \quad (3.51)$$

for all $0 < \epsilon \leq 1$. On the other hand, it is easy to check that there exists a uniform constant $\tilde{C}_{0,6}$ such that:

$$\int_{\tilde{M}} |\sqrt{-1}\Lambda_{\omega_\epsilon} F_{H_{S,\epsilon}(0)} - \lambda_{S,\epsilon} \text{Id}_S|_{H_{S,\epsilon}(0)} \frac{\omega_\epsilon^n}{n!} \leq \tilde{C}_{0,6}, \quad (3.52)$$

for all $0 < \epsilon \leq 1$. From the heat flow (3.47), one can check that

$$(\Delta_\omega - \frac{\partial}{\partial t}) |\sqrt{-1}\Lambda_{\omega_\epsilon} F_{\tilde{H}_{S,\epsilon}(t)} - \lambda_{S,\epsilon} \text{Id}_S|_{\tilde{H}_{S,\epsilon}(t)} \geq 0. \quad (3.53)$$

By the maximum principle and the uniform upper bounds on the heat kernels (3.8), we have

$$\max_{x \in \tilde{M}} |\sqrt{-1}\Lambda_{\omega_\epsilon} F_{\tilde{H}_{S,\epsilon}(t)} - \lambda_{S,\epsilon} \text{Id}_S|_{\tilde{H}_{S,\epsilon}(t)} \leq C_K \tilde{C}_{0,6} (1 + t^{-n}). \quad (3.54)$$

This implies that there is a uniform constant $\tilde{C}_{0,7}$ such that

$$\max_{x \in \tilde{M}} \log \{ \text{tr}(\tilde{H}_{S,\epsilon}^{-1}(1) K_{S,\epsilon}) + \text{tr}(K_{S,\epsilon}^{-1} \tilde{H}_{S,\epsilon}(1)) \} \leq \tilde{C}_{0,7}. \quad (3.55)$$

From (3.51) and (3.55), we know that there exists a constant $\tilde{C}_{0,8}$ such that

$$\max_{x \in \tilde{M}} \epsilon \cdot \left| \log \frac{\det(K_{S,\epsilon})}{\det(H_{S,\epsilon}(0))} \right| \leq \tilde{C}_{0,8}. \quad (3.56)$$

Let ξ be the defining section of the line bundle L with respect to the exceptional divisor $D \subset \pi^{-1}\Sigma_{alg}$. By the definition (3.5) of the Kähler metric η , we see that

$$\eta = \pi^* \omega - \delta \cdot \sqrt{-1} \partial \bar{\partial} \log |\xi|_{H_L}^2, \quad (3.57)$$

on $\tilde{M} \setminus D$. Using (3.43) and (3.48), we get

$$\begin{aligned} & \int_{\tilde{M}} \text{tr}(\varrho_{S,\epsilon} + \tilde{C}_{0,2} \text{Id}_S) \cdot \text{tr}(\sqrt{-1}\Lambda_{\omega_\epsilon} F_{H_{S,\epsilon}(0)}) \frac{\omega_\epsilon^n}{n!} \\ &= \int_{\tilde{M}} \text{tr}(\varrho_{S,\epsilon} + \tilde{C}_{0,2} \text{Id}_S) \cdot \text{tr}(\sqrt{-1} F_{H_{S,\epsilon}(0)}) \wedge \frac{\omega_\epsilon^{n-1}}{(n-1)!} \\ &\geq -\tilde{C}_{0,5} \int_{\tilde{M}} \text{tr}(\varrho_{S,\epsilon} + \tilde{C}_{0,2} \text{Id}_S) \cdot \eta \wedge \frac{\omega_\epsilon^{n-1}}{(n-1)!} \\ &= -\tilde{C}_{0,5} \int_{\tilde{M}} \text{tr}(\varrho_{S,\epsilon} + \tilde{C}_{0,2} \text{Id}_S) \cdot (\pi^* \omega - \delta \cdot \sqrt{-1} \partial \bar{\partial} \log |\xi|_{H_L}^2) \wedge \frac{\omega_\epsilon^{n-1}}{(n-1)!} \end{aligned} \quad (3.58)$$

Near the divisor D , there is a complex coordinate system $(U, (\tilde{z}^1, \dots, \tilde{z}^n))$ such that $D \cap U = \{\tilde{z}^n = 0\}$. So we can write locally:

$$|\xi|_{H_L}^2 = \phi |\tilde{z}^n|^{2m}, \quad (3.59)$$

where ϕ is a nowhere vanishing smooth function. Setting $\beta_1 = \sqrt{-1} \partial \bar{\partial} \log |\xi|_{H_L}^2$, we see that β_1 is 1-form with L_{loc}^1 coefficients on the entire \tilde{M} . By the residue formula, we have

$$\begin{aligned} & \int_{\tilde{M}} \text{tr}(\varrho_{S,\epsilon}) \cdot \sqrt{-1} \partial \bar{\partial} \log |\xi|_{H_L}^2 \wedge \frac{\omega_\epsilon^{n-1}}{(n-1)!} = \mathcal{T}_{d\beta_1}(\text{tr}(\varrho_{S,\epsilon}) \frac{\omega_\epsilon^{n-1}}{(n-1)!}) \\ &= d\mathcal{T}_{\beta_1}(\text{tr}(\varrho_{S,\epsilon}) \frac{\omega_\epsilon^{n-1}}{(n-1)!}) + 2m\pi \int_D \text{tr}(\varrho_{S,\epsilon}) \frac{\omega_\epsilon^{n-1}}{(n-1)!} \\ &= \mathcal{T}_{\beta_1}(d\text{tr}(\varrho_{S,\epsilon}) \wedge \frac{\omega_\epsilon^{n-1}}{(n-1)!}) + 2m\pi \int_D \text{tr}(\varrho_{S,\epsilon}) \frac{\omega_\epsilon^{n-1}}{(n-1)!}, \end{aligned} \quad (3.60)$$

where \mathcal{T}_β stands for the current with respect to the form β . Since $\text{tr}(\varrho_{S,\epsilon})$ is a smooth function on \tilde{M} , $D \subset \pi^{-1}\Sigma_{alg}$ and $\Sigma_{alg} \subset M$ is a subset of complex codimension greater than or equal to 2, we have

$$\int_D \text{tr}(\varrho_{S,\epsilon}) \frac{(\pi^*\omega)^{n-1}}{(n-1)!} = 0. \quad (3.61)$$

By the estimate (3.56), we obtain

$$\begin{aligned} & \int_D \text{tr}(\varrho_{S,\epsilon}) \frac{\omega_\epsilon^{n-1}}{(n-1)!} \\ &= \frac{1}{(n-1)!} \int_D \text{tr}(\varrho_{S,\epsilon}) ((\pi^*\omega)^{n-1} + \epsilon(\sum_{i=1}^{n-1} C_i^{n-1} \epsilon^{i-1} \eta^i \wedge (\pi^*\omega)^{n-1-i})) \\ &= \frac{1}{(n-1)!} \int_D \epsilon \text{tr}(\varrho_{S,\epsilon}) (\sum_{i=1}^{n-1} C_i^{n-1} \epsilon^{i-1} \eta^i \wedge (\pi^*\omega)^{n-1-i}) \\ &\geq -\tilde{C}_{0,9} \end{aligned} \quad (3.62)$$

for all $0 < \epsilon \leq 1$, where $\tilde{C}_{0,9}$ is a positive constant. Set $\beta_2 = \sqrt{-1} \log |\xi|_{H_L}^2$, which is L_{loc}^1 on the entire \tilde{M} . Using the residue formula again, we have

$$\begin{aligned} & \mathcal{T}_{\beta_1} (d\text{tr}(\varrho_{S,\epsilon}) \wedge \frac{\omega_\epsilon^{n-1}}{(n-1)!}) \\ &= \int_{\tilde{M}} \sqrt{-1} \partial \log |\xi|_{H_L}^2 \wedge \partial \text{tr}(\varrho_{S,\epsilon}) \wedge \frac{\omega_\epsilon^{n-1}}{(n-1)!} \\ &= \mathcal{T}_{d\beta_2} (\partial \text{tr}(\varrho_{S,\epsilon}) \wedge \frac{\omega_\epsilon^{n-1}}{(n-1)!}) \\ &= -d\mathcal{T}_{\beta_2} (\partial \text{tr}(\varrho_{S,\epsilon}) \wedge \frac{\omega_\epsilon^{n-1}}{(n-1)!}) - 2m\pi \int_D \partial \text{tr}(\varrho_{S,\epsilon}) \wedge \frac{\omega_\epsilon^{n-1}}{(n-1)!} \\ &= \int_{\tilde{M}} \log |\xi|_{H_L}^2 \sqrt{-1} \partial \bar{\partial} \text{tr}(\varrho_{S,\epsilon}) \wedge \frac{\omega_\epsilon^{n-1}}{(n-1)!} - 2m\pi \int_D \partial \text{tr}(\varrho_{S,\epsilon}) \wedge \frac{\omega_\epsilon^{n-1}}{(n-1)!}. \end{aligned} \quad (3.63)$$

Since D is a subset of complex codimension 1, we see

$$\int_D \partial \text{tr}(\varrho_{S,\epsilon}) \wedge \frac{\omega_\epsilon^{n-1}}{(n-1)!} = 0. \quad (3.64)$$

On the other hand, it is easy to check that:

$$\begin{aligned} & \int_{\tilde{M}} \log |\xi|_{H_L}^2 \sqrt{-1} \partial \bar{\partial} \text{tr}(\varrho_{S,\epsilon}) \wedge \frac{\omega_\epsilon^{n-1}}{(n-1)!} \\ &= \int_{\tilde{M}} \log |\xi|_{H_L}^2 \sqrt{-1} \partial \bar{\partial} \log \frac{\det(K_{S,\epsilon})}{\det(H_{S,\epsilon}(0))} \wedge \frac{\omega_\epsilon^{n-1}}{(n-1)!} \\ &= \int_{\tilde{M}} \log |\xi|_{H_L}^2 \sqrt{-1} \text{tr}(F_{H_{S,\epsilon}(0)} - F_{K_{S,\epsilon}}) \wedge \frac{\omega_\epsilon^{n-1}}{(n-1)!} \\ &= \int_{\tilde{M}} \log |\xi|_{H_L}^2 \{ \sqrt{-1} \text{tr}(F_{H_{S,\epsilon}(0)}) \wedge \frac{\omega_\epsilon^{n-1}}{(n-1)!} - \text{rank}(S) \lambda_{S,\epsilon} \frac{\omega_\epsilon^n}{n!} \} \\ &\geq -\tilde{C}_{0,10} \end{aligned} \quad (3.65)$$

for all $0 < \epsilon \leq 1$, where $\tilde{C}_{0,10}$ is a positive constant. Due to (3.58), (3.62), (3.63), (3.64) and (3.65), there is a constant $\tilde{C}_{0,11}$ such that

$$\int_{\tilde{M}} \operatorname{tr}(\varrho_{S,\epsilon}) \cdot \operatorname{tr}(\sqrt{-1}\Lambda_{\omega_\epsilon} F_{H_{S,\epsilon}(0)}) \frac{\omega_\epsilon^n}{n!} \geq -\tilde{C}_{0,11} \quad (3.66)$$

for all $0 < \epsilon \leq 1$. Then (3.44) and (3.66) imply (3.38).

Applying the Gauss-Codazzi equation, we know

$$\operatorname{tr}(\sqrt{-1}\Lambda_{\omega_\epsilon} F_{H_{S,\epsilon}(0)}) + \operatorname{tr}(\sqrt{-1}\Lambda_{\omega_\epsilon} F_{H_{Q,\epsilon}(0)}) = \operatorname{tr}(\sqrt{-1}\Lambda_{\omega_\epsilon} F_{\pi^* H_0}), \quad (3.67)$$

and

$$\begin{aligned} & \Delta_\epsilon \{ \operatorname{tr}(\varrho_{S,\epsilon}) + \operatorname{tr}(\varrho_{Q,\epsilon}) \} \\ &= \operatorname{tr}(\sqrt{-1}\Lambda_{\omega_\epsilon} F_{H_{S,\epsilon}(0)} - \lambda_{S,\epsilon} \operatorname{Id}_S) \\ & \quad + \operatorname{tr}(\sqrt{-1}\Lambda_{\omega_\epsilon} F_{H_{Q,\epsilon}(0)} - \lambda_{Q,\epsilon} \operatorname{Id}_Q) \\ &= \operatorname{tr}(\sqrt{-1}\Lambda_{\omega_\epsilon} F_{\pi^* H_0} - \lambda_{\tilde{\mathcal{E}},\epsilon} \operatorname{Id}_{\tilde{\mathcal{E}}}). \end{aligned} \quad (3.68)$$

By (3.68), (3.22), (3.19) and the uniform lower bounds of the Green functions (3.9), there is a constant $\tilde{C}_{0,12}$ such that

$$\max_{x \in \tilde{M}} |\operatorname{tr}(\varrho_{S,\epsilon}) + \operatorname{tr}(\varrho_{Q,\epsilon})| \leq \tilde{C}_{0,12}. \quad (3.69)$$

Then (3.45), (3.67), (3.69) and (3.66) imply (3.39). \square

Proposition 3.4. *Along the heat flow (3.10), we have*

$$\operatorname{tr}(\sqrt{-1}\Lambda_{\omega_\epsilon} (F_{H_{S,\epsilon}} - \gamma_\epsilon \wedge \gamma_\epsilon^*) - \lambda_{S,\epsilon} \operatorname{Id}_S) \leq v_\epsilon(t), \quad (3.70)$$

for all $0 < \epsilon \leq 1$, where $v_\epsilon(t) \geq 0$ satisfies: $\int_1^{+\infty} v(t) dt < \tilde{C}_{0,13}$, and $\tilde{C}_{0,13}$ is a uniform constant.

Proof. In [15], for any exact sequence of holomorphic bundles $0 \rightarrow S \rightarrow \tilde{\mathcal{E}} \rightarrow Q \rightarrow 0$, Donaldson has proved that

$$\begin{aligned} \mathcal{M}_{\tilde{\mathcal{E}},\epsilon}^0(H_0, H(t)) &= \mathcal{M}_{S,\epsilon}^0(H_{S,0}, H_S) + \mathcal{M}_{Q,\epsilon}^0(H_{Q,0}, H_Q) \\ & \quad + \|\gamma(t)\|_{L^2}^2 - \|\gamma(0)\|_{L^2}^2. \end{aligned} \quad (3.71)$$

By the equations (2.16) and (2.17), we have

$$\begin{aligned} & \int_{\tilde{M}} \log \det(H_{S,\epsilon}^{-1}(0) H_{S,\epsilon}(t)) \frac{\omega_\epsilon^n}{n!} \\ &= \int_0^t \frac{\partial}{\partial l} \int_{\tilde{M}} \log \det(H_{S,\epsilon}^{-1}(0) H_{S,\epsilon}(l)) \frac{\omega_\epsilon^n}{n!} dl \\ &= -2 \int_0^t \int_{\tilde{M}} \operatorname{tr}(\sqrt{-1}\Lambda_{\omega_\epsilon} (F_{H_{S,\epsilon}} - \gamma_\epsilon \wedge \gamma_\epsilon^*) - \lambda_{\tilde{\mathcal{E}},\epsilon} \operatorname{Id}_S) \frac{\omega_\epsilon^n}{n!} dl \\ &= -2 \int_0^t \int_{\tilde{M}} |\gamma_\epsilon(l)|_{H(l)}^2 \frac{\omega_\epsilon^n}{n!} dl - 2(\lambda_{S,\epsilon} - \lambda_{\tilde{\mathcal{E}},\epsilon}) \operatorname{rank}(S) t \operatorname{Vol}(\tilde{M}, \omega_\epsilon), \end{aligned} \quad (3.72)$$

and

$$\int_{\tilde{M}} \log \det(H_{Q,\epsilon}^{-1}(0) H_{Q,\epsilon}(t)) \frac{\omega_\epsilon^n}{n!} = 2 \int_0^t \int_{\tilde{M}} |\gamma_\epsilon(l)|_{H(l)}^2 dl - 2(\lambda_{Q,\epsilon} - \lambda_{\tilde{\mathcal{E}},\epsilon}) \operatorname{rank}(Q) t \operatorname{Vol}(\tilde{M}, \omega_\epsilon). \quad (3.73)$$

Then

$$\begin{aligned}
\mathcal{M}_{\tilde{\mathcal{E}},\epsilon}(H_0, H) &= \mathcal{M}_{S,\epsilon}(H_{S,\epsilon}(0), H_{S,\epsilon}(t)) + \mathcal{M}_{Q,\epsilon}(H_{Q,\epsilon}(0), H_{Q,\epsilon}(t)) \\
&+ \|\gamma_\epsilon(t)\|_{L^2(\omega_\epsilon)}^2 - \|\gamma(0)\|_{L^2(\omega_\epsilon)}^2 \\
&- 2(\lambda_{S,\epsilon} - \lambda_{\tilde{\mathcal{E}},\epsilon})^2 \text{rank}(S)t \text{Vol}(\tilde{M}, \omega_\epsilon) - 2(\lambda_{Q,\epsilon} - \lambda_{\tilde{\mathcal{E}},\epsilon})^2 \text{rank}(Q)t \text{Vol}(\tilde{M}, \omega_\epsilon) \\
&- 2(\lambda_{S,\epsilon} - \lambda_{Q,\epsilon}) \int_0^t \int_{\tilde{M}} |\gamma_\epsilon(l)|_{H(l)}^2 \frac{\omega_\epsilon^n}{n!} dl.
\end{aligned} \tag{3.74}$$

Furthermore, from (3.37), (3.34) and the Gauss-Codazzi equation (2.10), it follows that

$$\begin{aligned}
\mathcal{M}_{S,\epsilon}(H_{S,\epsilon}(0), H_{S,\epsilon}(t)) + \mathcal{M}_{Q,\epsilon}(H_{Q,\epsilon}(0), H_{Q,\epsilon}(t)) &= -\|\gamma_\epsilon(t)\|_{L^2(\omega_\epsilon)}^2 + \|\gamma(0)\|_{L^2(\omega_\epsilon)}^2 \\
&- 2(\lambda_{S,\epsilon} - \lambda_{Q,\epsilon}) \int_0^t \int_{\tilde{M}} |\gamma_\epsilon(l)|_{H(l)}^2 \frac{\omega_\epsilon^n}{n!} dl - 4 \int_0^t \|\partial_{H_{S,\epsilon}} \gamma_\epsilon + \gamma_\epsilon \partial_{H_{Q,\epsilon}}\|_{L^2(\omega_\epsilon)}^2(l) dl \\
&- 2 \int_0^t (\|\sqrt{-1} \Lambda_{\omega_\epsilon}(F_{H_{S,\epsilon}} - \gamma_\epsilon \wedge \gamma_\epsilon^*) - \lambda_{S,\epsilon} \text{Id}_S\|_{L^2(\omega_\epsilon)}^2)(l) dl \\
&- 2 \int_0^t (\|\sqrt{-1} \Lambda_{\omega_\epsilon}(F_{H_{Q,\epsilon}} - \gamma_\epsilon^* \wedge \gamma_\epsilon) - \lambda_{Q,\epsilon} \text{Id}_Q\|_{L^2(\omega_\epsilon)}^2)(l) dl.
\end{aligned} \tag{3.75}$$

Now, we set:

$$\begin{aligned}
\tilde{v}(\epsilon, t) &= 2\|\partial_{H_{S,\epsilon}} \gamma_\epsilon + \gamma_\epsilon \partial_{H_{Q,\epsilon}}\|_{L^2(\omega_\epsilon)}^2(t) + (\lambda_{S,\epsilon} - \lambda_{Q,\epsilon}) \|\gamma_\epsilon\|_{L^2(\omega_\epsilon)}^2(t) \\
&+ \|\sqrt{-1} \Lambda_{\omega_\epsilon}(F_{H_{S,\epsilon}} - \gamma_\epsilon \wedge \gamma_\epsilon^*) - \lambda_{S,\epsilon} \text{Id}_S\|_{L^2(\omega_\epsilon)}^2(t) \\
&+ \|\sqrt{-1} \Lambda_{\omega_\epsilon}(F_{H_{Q,\epsilon}} - \gamma_\epsilon^* \wedge \gamma_\epsilon) - \lambda_{Q,\epsilon} \text{Id}_Q\|_{L^2(\omega_\epsilon)}^2(t).
\end{aligned} \tag{3.76}$$

Combining the uniform lower bounds of the Donaldson's functional (3.38), (3.39) and the above equality (3.75), we obtain

$$\int_0^{+\infty} \tilde{v}(\epsilon, t) dt \leq \tilde{C}_{0,14} < +\infty, \tag{3.77}$$

for all $0 < \epsilon \leq 1$, where $\tilde{C}_{0,14}$ is a uniform positive constant.

On the other hand, along the heat flow (3.10) on $\tilde{\mathcal{E}}$, we have

$$(\Delta_\epsilon - \frac{\partial}{\partial t}) |\sqrt{-1} \Lambda_{\omega_\epsilon} F_{H_\epsilon(t)} - \lambda_{\tilde{\mathcal{E}},\epsilon} \text{Id}_{\tilde{\mathcal{E}}}|_{H_\epsilon(t)}^2 \geq 0. \tag{3.78}$$

By the uniform Sobolev inequality (3.6) for $(\tilde{M}, \omega_\epsilon)$ and the estimate of the heat kernel $\chi(x, y, t)$ by Cheng and Li ((2.9) in [9], or Theorem 3.2 in [6]), there is a positive constant $\tilde{C}_{0,15}$ such that

$$0 < \mathcal{K}_\epsilon(x, y, t) \leq \frac{1}{\text{Vol}(\tilde{M}, \omega_\epsilon)} + \tilde{C}_{0,15} t^{-n}, \tag{3.79}$$

for $0 < \epsilon \leq 1$. By the maximum principle, we have

$$\begin{aligned}
&|\sqrt{-1} \Lambda_{\omega_\epsilon} F_{H_\epsilon(t)} - \lambda_{\tilde{\mathcal{E}},\epsilon} \text{Id}_{\tilde{\mathcal{E}}}|_{H_\epsilon(t)}^2(x, t+s) \\
&\leq \int_{\tilde{M}} \mathcal{K}_\epsilon(x, y, s) |\sqrt{-1} \Lambda_{\omega_\epsilon} F_{H_\epsilon(t)} - \lambda_{\tilde{\mathcal{E}},\epsilon} \text{Id}_{\tilde{\mathcal{E}}}|_{H_\epsilon(t)}^2(y, t) \frac{\omega_\epsilon^n}{n!}(y) \\
&\leq (\frac{1}{\text{Vol}(\tilde{M}, \omega_\epsilon)} + \tilde{C}_{0,15} s^{-n}) \int_{\tilde{M}} |\sqrt{-1} \Lambda_{\omega_\epsilon} F_{H_\epsilon(t)} - \lambda_{\tilde{\mathcal{E}},\epsilon} \text{Id}_{\tilde{\mathcal{E}}}|_{H_\epsilon(t)}^2(y, t) \frac{\omega_\epsilon^n}{n!}(y),
\end{aligned} \tag{3.80}$$

for any $t > 0$, $s > 0$ and $0 < \epsilon \leq 1$.

Using the Gauss-Codazzi equation (2.10) again, and combining the formulas (3.34) and (3.80), we have

$$\begin{aligned}
& 2(\lambda_{S,\epsilon} - \lambda_{Q,\epsilon}) \operatorname{tr} (\sqrt{-1} \Lambda_{\omega_\epsilon} (F_{H_{S,\epsilon}} - \gamma_\epsilon \wedge \gamma_\epsilon^*) - \lambda_{S,\epsilon} \operatorname{Id}_S)(t) \\
& \leq |\sqrt{-1} \Lambda_{\omega_\epsilon} F_{H_\epsilon(t)} - \lambda_{\tilde{\mathcal{E}},\epsilon} \operatorname{Id}_{\tilde{\mathcal{E}}}|_{H_\epsilon(t)}^2 \\
& \quad - (\lambda_{S,\epsilon} - \lambda_{\tilde{\mathcal{E}},\epsilon})^2 \operatorname{rank}(S) - (\lambda_{Q,\epsilon} - \lambda_{\tilde{\mathcal{E}},\epsilon})^2 \operatorname{rank}(Q) \\
& \quad - 2(\lambda_{Q,\epsilon} - \lambda_{\mathcal{E},\epsilon}) \operatorname{tr} (\sqrt{-1} \Lambda_{\omega_\epsilon} F_{H_\epsilon(t)} - \lambda_{\mathcal{E},\epsilon} \operatorname{Id}_{\mathcal{E}}) \\
& \leq \left(\frac{1}{\operatorname{Vol}(\tilde{M}, \omega_\epsilon)} + \tilde{C}_{0,15} \left(\frac{t}{2} \right)^{-n} \right) \int_{\tilde{M}} |\sqrt{-1} \Lambda_{\omega_\epsilon} F_{H_\epsilon(\frac{t}{2})} - \lambda_{\tilde{\mathcal{E}},\epsilon} \operatorname{Id}_{\tilde{\mathcal{E}}}|_{H_\epsilon(\frac{t}{2})}^2 \\
& \quad - (\lambda_{S,\epsilon} - \lambda_{\tilde{\mathcal{E}},\epsilon})^2 \operatorname{rank}(S) - (\lambda_{Q,\epsilon} - \lambda_{\tilde{\mathcal{E}},\epsilon})^2 \operatorname{rank}(Q) + 2(\lambda_{\mathcal{E},\epsilon} - \lambda_{Q,\epsilon}) |f_\epsilon(t)| \\
& \leq \left(\frac{1}{\operatorname{Vol}(\tilde{M}, \omega_\epsilon)} + \tilde{C}_{0,15} \left(\frac{t}{2} \right)^{-n} \right) \tilde{v}(\epsilon, \frac{t}{2}) + 2(\lambda_{\mathcal{E},\epsilon} - \lambda_{Q,\epsilon}) |f_\epsilon(t)| \\
& \quad + \operatorname{Vol}(\tilde{M}, \omega_\epsilon) \tilde{C}_{0,15} \left(\frac{t}{2} \right)^{-n} \{ (\lambda_{S,\epsilon} - \lambda_{\mathcal{E},\epsilon})^2 \operatorname{rank}(S) + (\lambda_{Q,\epsilon} - \lambda_{\mathcal{E},\epsilon})^2 \operatorname{rank}(Q) \}.
\end{aligned} \tag{3.81}$$

Set

$$\begin{aligned}
v_\epsilon(t) &= (2(\lambda_{S,\epsilon} - \lambda_{Q,\epsilon}))^{-1} \left\{ \left(\frac{1}{\operatorname{Vol}(\tilde{M}, \omega_\epsilon)} + \tilde{C}_{0,15} \left(\frac{t}{2} \right)^{-n} \right) \tilde{v}(\epsilon, \frac{t}{2}) + 2(\lambda_{\mathcal{E},\epsilon} - \lambda_{Q,\epsilon}) \max_{x \in \tilde{M}} |f_\epsilon(x, t)| \right. \\
& \quad \left. + \operatorname{Vol}(\tilde{M}, \omega_\epsilon) \tilde{C}_{0,15} \left(\frac{t}{2} \right)^{-n} \{ (\lambda_{S,\epsilon} - \lambda_{\mathcal{E},\epsilon})^2 \operatorname{rank}(S) + (\lambda_{Q,\epsilon} - \lambda_{\mathcal{E},\epsilon})^2 \operatorname{rank}(Q) \} \right\}.
\end{aligned} \tag{3.82}$$

By the formulas (3.77), (3.81), (3.32) and note that $n > 1$, we see that $v(t)$ is the function which we need. \square

Using the above proposition, we can obtain a uniform local C^0 -bound on the rescaled metrics $\tilde{H}_S(t) = e^{2(\lambda_S - \lambda_E)t} H_S(t)$ and $\tilde{H}_Q(t) = e^{2(\lambda_Q - \lambda_E)t} H_Q(t)$.

Theorem 3.5. *Let $H(t)$ be the solution of the Hermitian-Yang-Mills flow (1.6) on the holomorphic bundle \mathcal{E} with initial metric H_0 , $H_S(t)$ and $H_Q(t)$ be the induced Hermitian metrics on $\mathcal{S}|_{M \setminus \Sigma_{alg}}$ and $\mathcal{Q}|_{M \setminus \Sigma_{alg}}$. Set $\hat{h}_S(t) = e^{2(\lambda_S - \lambda_E)t} H_S(0)^{-1} H_S(t)$ and $\hat{h}_Q(t) = e^{2(\lambda_Q - \lambda_E)t} H_Q(0)^{-1} H_Q(t)$. Then for any compact subset $U \subset M \setminus \Sigma_{alg}$ there exists a uniform constant \tilde{C}_U such that*

$$\operatorname{tr} \hat{h}_S(x, t) + \operatorname{tr} \hat{h}_S^{-1}(x, t) + \operatorname{tr} \hat{h}_Q(x, t) + \operatorname{tr} \hat{h}_Q^{-1}(x, t) \leq \tilde{C}_U, \tag{3.83}$$

for all $t \geq 0$ and $x \in U$.

Proof. In the following, we denote:

$$B_\Sigma(\delta) = \{x \in M | d(x, \Sigma_{alg}) < \delta\}, \tag{3.84}$$

where d is the distance function with respect to the Kähler metric ω . We can choose a small number δ_0 such that $U \subset M \setminus B_\Sigma(\delta_0)$. Since $\pi^* \omega$ is positive on $\tilde{M} \setminus \pi^{-1} \Sigma_{alg}$, there is a constant $C_{\delta_0^{-1}}$ such that

$$C_{\delta_0^{-1}}^{-1} \eta \leq \pi^* \omega \leq C_{\delta_0^{-1}} \eta \tag{3.85}$$

on $\pi^{-1}(M \setminus B_\Sigma(\frac{1}{2} \delta_0))$. Noting that the metrics $H_{S,\epsilon}(0)$ and $H_{Q,\epsilon}(0)$ are independent of ϵ , one checks that

$$-C_{U,1} \operatorname{Id}_S \leq \sqrt{-1} \Lambda_{\omega_\epsilon} F_{H_{S,\epsilon}(0)} \leq C_{U,1} \operatorname{Id}_S, \tag{3.86}$$

and

$$-C_{U,1}\text{Id}_Q \leq \sqrt{-1}\Lambda_{\omega_\epsilon} F_{H_{Q,\epsilon}(0)} \leq C_{U,1}\text{Id}_Q \quad (3.87)$$

on $\pi^{-1}(M \setminus B_\Sigma(\frac{1}{2}\delta_0))$, where $C_{U,1}$ is a uniform constant independent of ϵ . Then there is a uniform constant $C_{U,2}$ such that

$$\Delta_\epsilon \log(\text{tr } \tilde{h}_{S,\epsilon}(0) + \text{tr } \tilde{h}_{S,\epsilon}^{-1}(0)) \geq -C_{U,2} \quad (3.88)$$

and

$$\Delta_\epsilon \log(\text{tr } \tilde{h}_{Q,\epsilon}(0) + \text{tr } \tilde{h}_{Q,\epsilon}^{-1}(0)) \geq -C_{U,2} \quad (3.89)$$

on $\pi^{-1}(M \setminus B_\Sigma(\frac{1}{2}\delta_0))$, for all $0 < \epsilon \leq 1$. By the uniform Sobolev inequality (3.6), (3.88), (3.89) and Moser's iteration, we have the following mean-value inequalities, i.e. there exists a uniform constant $C_{U,3}$ such that

$$\sup_{\pi^{-1}(M \setminus B_\Sigma(\delta_0))} \log(\text{tr } \tilde{h}_{S,\epsilon}(0) + \text{tr } \tilde{h}_{S,\epsilon}^{-1}(0)) \leq C_{U,3} \int_{\tilde{M}} \log\{\text{tr } (\tilde{h}_{S,\epsilon}(0)) + \text{tr } (\tilde{h}_{S,\epsilon}^{-1}(0))\} \frac{\omega_\epsilon^n}{n!} \quad (3.90)$$

and

$$\sup_{\pi^{-1}(M \setminus B_\Sigma(\delta_0))} \log(\text{tr } \tilde{h}_{Q,\epsilon}(0) + \text{tr } \tilde{h}_{Q,\epsilon}^{-1}(0)) \leq C_{U,3} \int_{\tilde{M}} \log\{\text{tr } (\tilde{h}_{Q,\epsilon}(0)) + \text{tr } (\tilde{h}_{Q,\epsilon}^{-1}(0))\} \frac{\omega_\epsilon^n}{n!}. \quad (3.91)$$

From (3.19), we know that there is a uniform constant $C_{U,4}$ such that

$$\sup_{\pi^{-1}(M \setminus B_\Sigma(\delta_0))} \{\log(\text{tr } \tilde{h}_{S,\epsilon}(0) + \text{tr } \tilde{h}_{S,\epsilon}^{-1}(0)) + \log(\text{tr } \tilde{h}_{Q,\epsilon}(0) + \text{tr } \tilde{h}_{Q,\epsilon}^{-1}(0))\} \leq C_{U,4}, \quad (3.92)$$

for all $0 < \epsilon \leq 1$. Set

$$\hat{h}_{S,\epsilon}(t) = e^{2(\lambda_{S,\epsilon} - \lambda_{\tilde{\mathcal{E}},\epsilon})t} H_{S,\epsilon}(0)^{-1} H_{S,\epsilon}(t)$$

and

$$\hat{h}_{Q,\epsilon}(t) = e^{2(\lambda_{Q,\epsilon} - \lambda_{\tilde{\mathcal{E}},\epsilon})t} H_{Q,\epsilon}(0)^{-1} H_{Q,\epsilon}(t).$$

(3.17) and (3.92) imply that

$$\sup_{(x,t) \in \pi^{-1}(M \setminus B_\Sigma(\delta_0)) \times [0, +\infty)} (\text{tr } \hat{h}_{S,\epsilon}(x, t) + \text{tr } \hat{h}_{Q,\epsilon}^{-1}(x, t)) \leq C_{U,5}, \quad (3.93)$$

for all $0 < \epsilon \leq 1$, where $C_{U,5}$ is a uniform constant independent of ϵ . Due to (2.16), it holds that

$$\frac{\partial}{\partial t} \log(\det \hat{h}_{S,\epsilon}^{-1}) = 2\text{tr}(\sqrt{-1}\Lambda_{\omega_\epsilon}(F_{H_{S,\epsilon}} - \gamma_\epsilon \wedge \gamma_\epsilon^*) - \lambda_{S,\epsilon}\text{Id}_S). \quad (3.94)$$

By (3.22), the Gauss-Codazzi equation (2.10) and the maximum principle, we have

$$\sup_{(x,t) \in \tilde{M} \times [0, +\infty)} |\sqrt{-1}\Lambda_{\omega_\epsilon} F_{H_\epsilon(t)} - \lambda_{\tilde{\mathcal{E}},\epsilon} \text{Id}_{\tilde{\mathcal{E}}}^2|_{H_\epsilon(t)}^2 \leq \overline{C}_F \quad (3.95)$$

and

$$\sup_{(x,t) \in \tilde{M} \times [0, +\infty)} |\text{tr}(\sqrt{-1}\Lambda_{\omega_\epsilon}(F_{H_{S,\epsilon}} - \gamma_\epsilon \wedge \gamma_\epsilon^*) - \lambda_{S,\epsilon}\text{Id}_S)| \leq \overline{C}_F, \quad (3.96)$$

where \overline{C}_F is a uniform constant independent of ϵ . Then (3.70) implies that there exists a uniform constant $C_{U,6}$ such that

$$\sup_{(x,t) \in \tilde{M} \times [0, +\infty)} \log(\det \hat{h}_{S,\epsilon}^{-1}(x, t)) \leq C_{U,6}, \quad (3.97)$$

for all $0 < \epsilon \leq 1$. Combining (3.33), (3.32) and (3.97), we see that

$$\det(\hat{h}_{S,\epsilon}(t)) \det(\hat{h}_{Q,\epsilon}(t)) = \det(H_0^{-1} H_\epsilon(t)), \quad (3.98)$$

and then

$$\begin{aligned} \log \det(\hat{h}_{Q,\epsilon}(x, t)) &= \log \det(\hat{h}_{S,\epsilon}^{-1}(x, t)) + \log \det(H_0^{-1}H_\epsilon(x, t)) \\ &\leq C_{U,7} \end{aligned} \quad (3.99)$$

for all $(x, t) \in \tilde{M} \times [0, +\infty)$ and $0 < \epsilon \leq 1$, where $C_{U,7}$ is a uniform constant independent of ϵ . From (3.100), (3.97) and (3.99), it is easy to see that there exists a constant $C_{U,8}$ such that

$$\sup_{(x,t) \in \pi^{-1}(M \setminus B_\Sigma(\delta_0)) \times [0, +\infty)} (\text{tr } \hat{h}_{S,\epsilon} + \text{tr } \hat{h}_{S,\epsilon}^{-1} + \text{tr } \hat{h}_{Q,\epsilon} + \text{tr } \hat{h}_{Q,\epsilon}^{-1})(x, t) \leq C_{U,8} \quad (3.100)$$

for all $0 < \epsilon \leq 1$. Taking the limit $\epsilon \rightarrow 0$, we obtain (3.83). \square

4. UNIFORM LOCAL C^1 -ESTIMATE

Using the above local C^0 -estimate of the rescaled metrics $\tilde{H}_S(t)$ and $\tilde{H}_Q(t)$, we can control the L_{loc}^∞ -norm of $|G(t)|_{H(t)}$. In fact, we have the following proposition.

Proposition 4.1. *Let $H(t)$ be the solution of the heat flow (1.6) with initial metric H_0 on $(E, \bar{\partial}_E)$, and $G(t) \in \Gamma(\mathcal{S} \otimes \mathcal{Q}^*|_{M \setminus \Sigma_{alg}})$ be the section defined in (2.23). Then for any compact subset $U \subset M \setminus \Sigma_{alg}$, there exists a constant $C_{G,U}$ such that*

$$\int_U |G(t)|_{H(t)}^2 \frac{\omega^n}{n!} \leq C_{G,U} \quad (4.1)$$

for any $t \geq 0$. Furthermore, there exists a constant $\tilde{C}_{G,U}$ such that

$$\sup_{(x,t) \in U \times [0, +\infty)} |G(t)|_{H(t)}^2(x) \leq \tilde{C}_{G,U}. \quad (4.2)$$

Proof. Let δ be small enough so that $U \subset M \setminus B_\Sigma(\delta)$, and φ be a nonnegative cut-off function satisfying:

$$\varphi(x) = \begin{cases} 1, & x \in M \setminus B_\Sigma(\delta), \\ 0, & x \in B_\Sigma(\frac{1}{2}\delta). \end{cases} \quad (4.3)$$

Direct calculations yield

$$\begin{aligned} &\frac{\partial}{\partial t} \int_M \varphi^2 e^{-2(\lambda_S - \lambda_Q)t} |G(t)|_{H(0)}^2 \frac{\omega^n}{n!} \\ &= \int_M 2\varphi^2 e^{-2(\lambda_S - \lambda_Q)t} \text{Re} \left\langle \frac{\partial G}{\partial t}, G(t) \right\rangle_{H(0)} \frac{\omega^n}{n!} \\ &\quad - 2(\lambda_S - \lambda_Q) \int_M \varphi^2 e^{-2(\lambda_S - \lambda_Q)t} |G(t)|_{H(0)}^2 \frac{\omega^n}{n!} \\ &\leq 2 \left\{ \int_M \varphi^2 e^{-2(\lambda_S - \lambda_Q)t} \left| \frac{\partial G}{\partial t} \right|_{H(0)}^2 \frac{\omega^n}{n!} \right\}^{\frac{1}{2}} \left\{ \int_M \varphi^2 e^{-2(\lambda_S - \lambda_Q)t} |G(t)|_{H(0)}^2 \frac{\omega^n}{n!} \right\}^{\frac{1}{2}} \\ &\quad - 2(\lambda_S - \lambda_Q) \int_M \varphi^2 e^{-2(\lambda_S - \lambda_Q)t} |G(t)|_{H(0)}^2 \frac{\omega^n}{n!} \\ &\leq \frac{1}{2(\lambda_S - \lambda_Q)} \int_M \varphi^2 e^{-2(\lambda_S - \lambda_Q)t} \left| \frac{\partial G}{\partial t} \right|_{H(0)}^2 \frac{\omega^n}{n!}. \end{aligned} \quad (4.4)$$

For any section $\theta \in \Gamma(\mathcal{S} \otimes \mathcal{Q}^*)$, by the local C^0 -estimate (3.83), we know that there is a uniform constant $C_{\delta^{-1},1}$ such that

$$\begin{aligned} |\theta|_{H(0)}^2(x) &= \text{tr}(\theta \cdot H_Q^{-1}(0) \bar{\theta}^T H_S(0))(x) \\ &\leq C_{\delta^{-1},1} \text{tr}(\theta \cdot e^{-2(\lambda_Q - \lambda_E)t} H_Q^{-1}(t) \bar{\theta}^T e^{2(\lambda_S - \lambda_E)t} H_S(t))(x) \\ &= e^{2(\lambda_S - \lambda_Q)t} C_{\delta^{-1},1} \text{tr}(\theta \cdot H_Q^{-1}(t) \bar{\theta}^T H_S(t))(x) \\ &= C_{\delta^{-1},1} e^{2(\lambda_S - \lambda_Q)t} |\theta|_{H(t)}^2(x) \end{aligned} \quad (4.5)$$

and

$$|\theta|_{H(0)}^2(x) \geq C_{\delta^{-1},1}^{-1} e^{2(\lambda_S - \lambda_Q)t} |\theta|_{H(t)}^2(x) \quad (4.6)$$

for all $(x, t) \in M \setminus B_\Sigma(\frac{1}{2}\delta) \times [0, +\infty)$. Using (4.4), (4.5), (4.6) (2.24) and (3.77), we get

$$\begin{aligned} &\int_M \varphi^2 |G(t)|_{H(t)}^2 \frac{\omega^n}{n!} \\ &\leq C_{\delta^{-1},1} \int_M \varphi^2 e^{-2(\lambda_S - \lambda_Q)t} |G(t)|_{H(0)}^2 \frac{\omega^n}{n!} \\ &= C_{\delta^{-1},1} \int_0^t \frac{\partial}{\partial l} \int_M \varphi^2 e^{-2(\lambda_S - \lambda_Q)l} |G(l)|_{H(0)}^2 \frac{\omega^n}{n!} dl \\ &\leq C_{\delta^{-1},1} \frac{1}{2(\lambda_S - \lambda_Q)} \int_M \varphi^2 |\Lambda_\omega(\partial_{H_S(t)} \gamma(t) + \gamma(t) \partial_{H_Q(t)})|_{H(t)}^2 \frac{\omega^n}{n!} \\ &\leq C_{\delta^{-1},2}, \end{aligned} \quad (4.7)$$

for all $t \in [0, +\infty)$. It is easy to see that (4.7) implies (4.1).

Since $\bar{\partial}_{S \otimes Q^*} \gamma_0 = 0$, for any point $P \in U$, we have a domain $B_P(4R)$ and a local section $G_0 \in \Gamma(B_P(4R); S \otimes Q^*)$ such that $\gamma_0 = \bar{\partial}_{S \otimes Q^*} G_0$, where $R < \frac{1}{8}\delta$. By (2.44), we have

$$(\Delta - \frac{\partial}{\partial t})|G + G_0|_{H(t)}^2 \geq 0 \quad (4.8)$$

on $B_P(4R)$. Applying the parabolic mean value inequality (Theorem 14.5. in [33], or [34]), we obtain

$$\sup_{B_P(\frac{R}{2}) \times [t_0 + \frac{R^2}{8}, t_0 + \frac{R^2}{4}]} |G + G_0|_{H(t)}^2 \leq C_1 \int_{\frac{R^2}{16}}^{\frac{R^2}{2}} \int_{B_P(R)} |G + G_0|_{H(s+t_0)}^2 dv_\omega ds \quad (4.9)$$

for any $t_0 > 0$, where constant C_1 depends only on $\dim M$, lower bound of Ricci curvature and R^{-1} . As in (4.6), we know $|G_0|_{H(t)}$ is uniformly bounded on $B_P(2R) \times [0, +\infty)$. From (4.1), we see $|G|_{H(t)}$ is also uniformly bounded on $B_P(\frac{R}{2}) \times [0, +\infty)$. Since point P is arbitrary and U is compact, we can obtain a uniform bound for $|G|_{H(t)}$ on $U \times [0, +\infty)$, i.e. the inequality (4.2) is valid. \square

Set

$$T_S(t) = D_{(H_S(t), \bar{\partial}_S)} - D_{(H_{0,S}, \bar{\partial}_S)} = h_S^{-1} \partial_{H_{0,S}} h_S = (\partial_{H_S(t)} h_S) h_S^{-1} \quad (4.10)$$

and

$$T_Q(t) = D_{(H_Q(t), \bar{\partial}_Q)} - D_{(H_{0,Q}, \bar{\partial}_Q)} = h_S^{-1} \partial_{H_{0,Q}} h_Q = (\partial_{H_Q(t)} h_Q) h_Q^{-1}, \quad (4.11)$$

where $h_S(t) = H_{0,S}^{-1}H_S(t)$ and $h_Q(t) = H_{0,Q}^{-1}H_Q(t)$. By the definition, we have

$$\begin{aligned}\frac{\partial}{\partial t}T_S &= \frac{\partial}{\partial t}(H_S^{-1}(t)\partial H_S(t)) \\ &= \partial_{H_S(t)}(H_S^{-1}(t)\frac{\partial H_S(t)}{\partial t}),\end{aligned}\tag{4.12}$$

and

$$\begin{aligned}\frac{\partial}{\partial t}|T_S|_{H_S}^2 &= \frac{\partial}{\partial t}g^{k\bar{l}}\text{tr}(T_S(\partial_k)H_S^{-1}\overline{T_S(\partial_l)^T}H_S) \\ &= 2\text{Re}\{g^{k\bar{l}}\text{tr}(\frac{\partial T_S}{\partial t}(\partial_k)H_S^{-1}\overline{T_S(\partial_l)^T}H_S)\} \\ &\quad - g^{k\bar{l}}\text{tr}(T_S(\partial_k)H_S^{-1}\frac{\partial H_S}{\partial t}H_S^{-1}\overline{T_S(\partial_l)^T}H_S) \\ &\quad + g^{k\bar{l}}\text{tr}(T_S(\partial_k)H_S^{-1}\overline{T_S(\partial_l)^T}H_SH_S^{-1}\frac{\partial H_S}{\partial t}) \\ &= 2\text{Re}\langle\partial_{H_S}(H_S^{-1}\frac{\partial H_S}{\partial t}), T_S\rangle_{H_S(t)} \\ &\quad + \text{Re}\langle[H_S^{-1}\frac{\partial H_S}{\partial t}, T_S], T_S\rangle_{H_S(t)}.\end{aligned}\tag{4.13}$$

Now we calculate the Laplacian of $|T_S|_{H_S}^2$. In the following we choose the normal complex coordinates $\{z^1, \dots, z^m\}$ centered at the considering point, and denote $\frac{\partial}{\partial z^i}$ ($\frac{\partial}{\partial \bar{z}^j}$) by ∂_i ($\bar{\partial}_j$) for simplicity. Direct calculations yield

$$\begin{aligned}\Delta|T_S|_{H_S}^2 &= 2g^{i\bar{j}}\partial_i\bar{\partial}_j|T_S|_{H_S}^2 \\ &= 2\text{Re}\{g^{i\bar{j}}\langle\nabla_{\partial_i}^{H_S(t)}\nabla_{\bar{\partial}_j}^{H_S(t)}T_S + \nabla_{\bar{\partial}_j}^{H(t)}\nabla_{\partial_i}^{H(t)}T_S, T_S\rangle_{H_S(t)}\} \\ &\quad + 2|\nabla_{H_S(t)}T_S|_{H_S(t)}^2,\end{aligned}\tag{4.14}$$

$$\begin{aligned}\nabla_{\partial_i}^{H_S(t)}\nabla_{\bar{\partial}_j}^{H_S(t)}T_S(\partial_k) &= D_{\partial_i}^{H_S(t)}(\nabla_{\bar{\partial}_j}^{H_S(t)}T_S(\partial_k)) - (\nabla_{\bar{\partial}_j}^{H_S(t)}T_S)(\nabla_{\partial_i}\partial_k) \\ &= D_{\partial_i}^{H_S(t)}(F_{H_{0,S}}(\partial_k, \bar{\partial}_j) - F_{H_S}(\partial_k, \bar{\partial}_j)) \\ &= D_{\partial_i}^{H_S(t)}(F_{H_{0,S}}(\partial_k, \bar{\partial}_j)) - D_{\partial_k}^{H_S(t)}(F_{H_S}(\partial_i, \bar{\partial}_j)),\end{aligned}\tag{4.15}$$

where we have used the equality $F_{H_S} = F_{H_{0,S}} + \bar{\partial}_S(T_S)$, the Bianchi identity $(\nabla_{\partial_i}^{H_S(t)} F_{H_S})(\partial_k, \bar{\partial}_j) = (\nabla_{\partial_k}^{H_S(t)} F_{H_S})(\partial_i, \bar{\partial}_j)$, and $\nabla_{\partial_i} \partial_k = 0$ at the considering point. Furthermore,

$$\begin{aligned}
\nabla_{\bar{\partial}_j}^{H_S(t)} \nabla_{\partial_i}^{H_S(t)} T_S(\partial_k) &= \nabla_{\bar{\partial}_j}^{H_S(t)} (D_{\partial_i}^{H_S(t)} (T_S(\partial_k)) - T_S(\nabla_{\partial_i} \partial_k)) \\
&= \nabla_{\bar{\partial}_j}^{H_S(t)} D_{\partial_i}^{H_S(t)} (T_S(\partial_k)) - T_S(\nabla_{\bar{\partial}_j} \nabla_{\partial_i} \partial_k) \\
&= D_{\partial_i}^{H_S(t)} \nabla_{\bar{\partial}_j}^{H_S(t)} (T_S(\partial_k)) + \langle Rm(\partial_i, \bar{\partial}_j) \partial_k, \bar{\partial}_s \rangle g^{l\bar{s}} T_S(\partial_l) \\
&\quad - F_{H_S}(\partial_i, \bar{\partial}_j) \circ T_S(\partial_k) + T_S(\partial_k) \circ F_{H_S}(\partial_i, \bar{\partial}_j) \\
&= D_{\partial_i}^{H_S(t)} (F_{H_{0,S}}(\partial_k, \bar{\partial}_j) - F_{H_S}(\partial_k, \bar{\partial}_j)) \\
&\quad + \langle Rm(\partial_i, \bar{\partial}_j) \partial_k, \bar{\partial}_s \rangle g^{l\bar{s}} T_S(\partial_l) \\
&\quad - F_{H_S}(\partial_i, \bar{\partial}_j) \circ T_S(\partial_k) + T_S(\partial_k) \circ F_{H_S}(\partial_i, \bar{\partial}_j) \\
&= D_{\partial_i}^{H_S(t)} (F_{H_{0,S}}(\partial_k, \bar{\partial}_j)) - D_{\partial_k}^{H_S(t)} (F_{H_S}(\partial_i, \bar{\partial}_j)) \\
&\quad + \langle Rm(\partial_i, \bar{\partial}_j) \partial_k, \bar{\partial}_s \rangle g^{l\bar{s}} T_S(\partial_l) \\
&\quad - F_{H_S}(\partial_i, \bar{\partial}_j) \circ T_S(\partial_k) + T_S(\partial_k) \circ F_{H_S}(\partial_i, \bar{\partial}_j).
\end{aligned} \tag{4.16}$$

Combining the above equalities and recalling $D_{\partial_i}^{H_S(t)} - D_{\partial_i}^{H_{0,S}} = T_S(\partial_i)$, we get

$$\begin{aligned}
\Delta |T_S|_{H_S}^2 &= 2|\nabla^{H_S(t)} T_S|_{H_S(t)}^2 + 2Ric_\omega(\partial_k, \bar{\partial}_s) g^{k\bar{i}} g^{l\bar{s}} \text{tr}(T_S(\partial_l) H_S^{-1} \overline{(T_S(\partial_i))^T} H_S) \\
&\quad - 2Re\langle [\sqrt{-1}\Lambda_\omega F_{H_S}, T_S], T_S \rangle_{H_S(t)} \\
&\quad - 4Re\langle \partial_{H_S}(\sqrt{-1}\Lambda_\omega F_{H_S}), T_S \rangle_{H_S(t)} \\
&\quad + 4Re\{g^{i\bar{j}} g^{k\bar{l}} \langle [T_S(\partial_i), F_{H_{0,S}}(\partial_k, \bar{\partial}_j)], T_S(\partial_l) \rangle_{H_S(t)}\} \\
&\quad + 4Re\langle \partial_{H_{0,S}}(\sqrt{-1}\Lambda_\omega F_{H_{0,S}}), T_S \rangle_{H_S(t)}.
\end{aligned} \tag{4.17}$$

From (2.16), (4.13) and (4.17), it follows that

$$\begin{aligned}
&(\Delta - \frac{\partial}{t}) |T_S|_{H_S}^2 \\
&= 2|\nabla^{H_S(t)} T_S|_{H_S(t)}^2 + 2Ric_\omega(\partial_k, \bar{\partial}_s) g^{k\bar{i}} g^{l\bar{s}} \text{tr}(T_S(\partial_l) H_S^{-1} \overline{(T_S(\partial_i))^T} H_S) \\
&\quad - 2Re\langle [\sqrt{-1}\Lambda_\omega(\gamma \wedge \gamma^*), T_S], T_S \rangle_{H_S(t)} \\
&\quad - 4Re\langle \partial_{H_S}(\sqrt{-1}\Lambda_\omega(\gamma \wedge \gamma^*)), T_S \rangle_{H_S(t)} \\
&\quad + 4Re\{g^{i\bar{j}} g^{k\bar{l}} \langle [T_S(\partial_i), F_{H_{0,S}}(\partial_k, \bar{\partial}_j)], T_S(\partial_l) \rangle_{H_S(t)}\} \\
&\quad + 4Re\langle \partial_{H_{0,S}}(\sqrt{-1}\Lambda_\omega F_{H_{0,S}}), T_S \rangle_{H_S(t)},
\end{aligned} \tag{4.18}$$

and similarly

$$\begin{aligned}
&(\Delta - \frac{\partial}{t}) |T_Q|_{H_Q}^2 \\
&= 2|\nabla^{H_Q(t)} T_Q|_{H_Q(t)}^2 + 2Ric_\omega(\partial_k, \bar{\partial}_s) g^{k\bar{i}} g^{l\bar{s}} \text{tr}(T_Q(\partial_l) H_Q^{-1} \overline{(T_Q(\partial_i))^T} H_Q) \\
&\quad - 2Re\langle [\sqrt{-1}\Lambda_\omega(\gamma^* \wedge \gamma), T_Q], T_Q \rangle_{H_Q(t)} \\
&\quad - 4Re\langle \partial_{H_Q}(\sqrt{-1}\Lambda_\omega(\gamma^* \wedge \gamma)), T_Q \rangle_{H_Q(t)} \\
&\quad + 4Re\{g^{i\bar{j}} g^{k\bar{l}} \langle [T_Q(\partial_i), F_{H_{0,Q}}(\partial_k, \bar{\partial}_j)], T_Q(\partial_l) \rangle_{H_Q(t)}\} \\
&\quad + 4Re\langle \partial_{H_{0,Q}}(\sqrt{-1}\Lambda_\omega F_{H_{0,Q}}), T_Q \rangle_{H_Q(t)}.
\end{aligned} \tag{4.19}$$

Proposition 4.2. *Let $H(t)$ be the solution of the heat flow (1.6) with initial metric H_0 on $(E, \bar{\partial}_E)$, let $T_S(t)$ and $T_Q(t)$ be defined by (4.10) and (4.11), and $\gamma(t)$ be the second fundamental form. Then for any compact subset $U \subset M \setminus \Sigma_{alg}$, there exists a constant $\tilde{C}_{1,U}$ such that*

$$\sup_{(x,t) \in U \times [0, +\infty)} (|T_S(t)|_{H_S(t)}^2 + |T_Q(t)|_{H_Q(t)}^2 + |\gamma(t)|_{H_{S,Q}(t)}^2)(x, t) \leq \tilde{C}_{1,U}. \quad (4.20)$$

Proof. Let δ be small enough so that $U \subset M \setminus B_\Sigma(\delta)$. Since the second fundamental form $\bar{\partial}_{S \otimes Q^*} \gamma_0 = 0$, for any point $P \in U$, we have a domain $B_P(4r)$ and a local section $G_0 \in \Gamma(B_P(4r); S \otimes Q^*)$ such that $\gamma_0 = \bar{\partial}_{S \otimes Q^*} G_0$, where $r < \frac{1}{8}\delta$. Set

$$\rho_0 = \begin{pmatrix} \text{Id}_S & -G_0 \\ 0 & \text{Id}_Q \end{pmatrix}. \quad (4.21)$$

It is easy to check that

$$\begin{aligned} & \begin{pmatrix} \text{Id}_S & G_0 \\ 0 & \text{Id}_Q \end{pmatrix} \begin{pmatrix} \bar{\partial}_S & \gamma_0 \\ 0 & \bar{\partial}_Q \end{pmatrix} \begin{pmatrix} \text{Id}_S & -G_0 \\ 0 & \text{Id}_Q \end{pmatrix} \\ &= \begin{pmatrix} \bar{\partial}_S & \gamma_0 - \bar{\partial}_S \circ G_0 + G_0 \circ \bar{\partial}_Q \\ 0 & \bar{\partial}_Q \end{pmatrix}. \end{aligned} \quad (4.22)$$

So there exists a local bundle isomorphism $\rho_0 : S \oplus Q \rightarrow S \oplus Q$ such that

$$\rho_0^* \begin{pmatrix} \bar{\partial}_S & \gamma_0 \\ 0 & \bar{\partial}_Q \end{pmatrix} = \begin{pmatrix} \bar{\partial}_S & 0 \\ 0 & \bar{\partial}_Q \end{pmatrix}, \quad (4.23)$$

and locally

$$(f_{H_0} \circ \rho_0)^* (\bar{\partial}_E) = \begin{pmatrix} \bar{\partial}_S & 0 \\ 0 & \bar{\partial}_Q \end{pmatrix}. \quad (4.24)$$

Define a local Hermitian metric $\bar{H}_{0,E}$ on E by

$$\bar{H}_{0,E} \doteq (\rho_0^{-1} \circ f_{H_0}^{-1})^* (H_{0,S} \oplus H_{0,Q}), \quad (4.25)$$

and set

$$T_E(t) = D_{(H(t), \bar{\partial}_E)} - D_{(\bar{H}_{0,E}, \bar{\partial}_E)}. \quad (4.26)$$

Now we get

$$(f_{H_0} \circ \rho_0)^* D_{(\bar{H}_{0,E}, \bar{\partial}_E)} = \begin{pmatrix} \bar{\partial}_S + \partial_{H_{0,S}} & 0 \\ 0 & \bar{\partial}_Q + \partial_{H_{0,Q}} \end{pmatrix}, \quad (4.27)$$

$$(\rho_0^{-1} \circ f_{H_0}^{-1} \circ f_{H(t)}) = \begin{pmatrix} \text{Id}_S & G + G_0 \\ 0 & \text{Id}_Q \end{pmatrix}, \quad (4.28)$$

$$\begin{aligned} & (\rho_0^{-1} \circ f_{H_0}^{-1} \circ f_{H(t)})^* \begin{pmatrix} \bar{\partial}_S + \partial_{H_{0,S}} & 0 \\ 0 & \bar{\partial}_Q + \partial_{H_{0,Q}} \end{pmatrix} \\ &= \begin{pmatrix} \text{Id}_S & -G - G_0 \\ 0 & \text{Id}_Q \end{pmatrix} \begin{pmatrix} \bar{\partial}_S + \partial_{H_{0,S}} & 0 \\ 0 & \bar{\partial}_Q + \partial_{H_{0,Q}} \end{pmatrix} \begin{pmatrix} \text{Id}_S & G + G_0 \\ 0 & \text{Id}_Q \end{pmatrix} \\ &= \begin{pmatrix} \bar{\partial}_S + \partial_{H_{0,S}} & (\bar{\partial}_{S \otimes Q^*} + \partial_{(H_{0,S}, H_{0,Q})})(G + G_0) \\ 0 & \bar{\partial}_Q + \partial_{H_{0,Q}} \end{pmatrix} \\ &= \begin{pmatrix} \bar{\partial}_S + \partial_{H_{0,S}} & \gamma(t) + \partial_{(H_{0,S}, H_{0,Q})}(G + G_0) \\ 0 & \bar{\partial}_Q + \partial_{H_{0,Q}} \end{pmatrix}, \end{aligned} \quad (4.29)$$

and

$$\begin{aligned}
& f_{H(t)}^*(T_E(t)) \\
&= f_{H(t)}^*(D_{(H(t), \bar{\partial}_E)} - D_{(\bar{H}_{0,E}, \bar{\partial}_E)}) \\
&= f_{H(t)}^*(D_{(H(t), \bar{\partial}_E)}) - (\rho_0^{-1} \circ f_{H_0}^{-1} \circ f_{H(t)})^* \circ (f_{H_0} \circ \rho_0)^*(D_{(\bar{H}_{0,E}, \bar{\partial}_E)}) \\
&= \begin{pmatrix} \bar{\partial}_S + \partial_{H_S(t)} & \gamma(t) \\ -\gamma^*(t) & \bar{\partial}_Q + \partial_{H_Q(t)} \end{pmatrix} - \begin{pmatrix} \bar{\partial}_S + \partial_{H_{0,S}} & \gamma(t) + \partial_{(H_{0,S}, H_{0,Q})}(G + G_0) \\ 0 & \bar{\partial}_Q + \partial_{H_{0,Q}} \end{pmatrix} \\
&= \begin{pmatrix} T_S(t) & -\partial_{(H_{0,S}, H_{0,Q})}(G + G_0) \\ -\gamma^*(t) & T_Q(t) \end{pmatrix},
\end{aligned} \tag{4.30}$$

where ∂ denotes the $(1,0)$ part of the Chern connection, and $T_S = \partial_{H_S(t)} - \partial_{H_{0,S}}$, $T_Q = \partial_{H_Q(t)} - \partial_{H_{0,Q}}$. Then it holds that

$$\begin{aligned}
|T_E(t)|_{H(t)}^2 &= |f_{H(t)}^*(T_E(t))|_{f_{H(t)}^*(H(t))}^2 \\
&= \left| \begin{pmatrix} T_S(t) & -\partial_{(H_{0,S}, H_{0,Q})}(G + G_0) \\ -\gamma^*(t) & T_Q(t) \end{pmatrix} \right|_{f_{H(t)}^*(H(t))}^2 \\
&= |T_S(t)|_{H_S(t)}^2 + |T_Q(t)|_{H_Q(t)}^2 \\
&\quad + |\gamma(t)|_{H_{S,Q}(t)}^2 + |\partial_{(H_{0,S}, H_{0,Q})}(G + G_0)|_{H_{S,Q}(t)}^2.
\end{aligned} \tag{4.31}$$

On the other hand, calculating in the same way as that in (4.18), we have the following local parabolic inequality

$$\begin{aligned}
& \left(\Delta - \frac{\partial}{\partial t}\right) |T_E|_{H_E}^2 \\
&= 2|\nabla^{H_E(t)} T_E|_{H_E(t)}^2 + 2Ric_\omega(\partial_k, \bar{\partial}_s) g^{k\bar{i}} g^{l\bar{s}} \text{tr}(T_E(\partial_l) H_E^{-1} \overline{(T_E(\partial_i))^T} H_E) \\
&\quad + 4Re\{g^{i\bar{j}} g^{k\bar{l}} \langle [T_E(\partial_i), F_{\bar{H}_{0,E}}(\partial_k, \bar{\partial}_j)], T_E(\partial_l) \rangle_{H_E(t)}\} \\
&\quad + 4Re\langle \partial_{K_E}(\sqrt{-1}\Lambda_\omega F_{\bar{H}_{0,E}}), T_E \rangle_{H_E(t)},
\end{aligned} \tag{4.32}$$

and then

$$\begin{aligned}
& \left(\Delta - \frac{\partial}{\partial t}\right) |T_E|_{H_E(t)}^2 \\
&\geq 2|\nabla^{H_E(t)} T_E|_{H_E(t)}^2 - c_1 |T_E|_{H_E(t)}^2 - c_2
\end{aligned} \tag{4.33}$$

on $B_P(2r)$, where constants c_1 and c_2 depend only on $\max_{\overline{B_P(2r)}} |F_{\bar{H}_{0,E}}|$ and the lower bound of Ricci curvature of (M, ω) . Let φ_1, φ_2 be nonnegative cut-off functions satisfying

$$\varphi_1(x) = \begin{cases} 1, & x \in B_P(\frac{r}{4}), \\ 0, & x \in M \setminus B_P(\frac{r}{2}); \end{cases} \tag{4.34}$$

$$\varphi_2(x) = \begin{cases} 1, & x \in B_P(\frac{r}{2}), \\ 0, & x \in M \setminus B_P(r); \end{cases} \tag{4.35}$$

and

$$|d\varphi_i|_\omega^2, |\Delta\varphi_i| \leq \frac{C}{r^2} \tag{4.36}$$

for $i = 1, 2$. We consider the following test function:

$$\zeta_1(\cdot, t) = \varphi_1^2 |T_E|_{H_E(t)}^2 + a\varphi_2^2 (|G + G_0|_{H(t)}^2 + \text{tr} \hat{h}_S(t) + \text{tr} \hat{h}_Q^{-1}(t)), \tag{4.37}$$

where constant a will be chosen large enough.

Let $\zeta_1(q, t_0) = \max_{M \times [0, t_1]} \zeta_1$, by the definition of φ_i and the locally uniform estimates of $|G + G_0|_{H(t)}^2$, $\text{tr } \hat{h}_S(t)$, $\text{tr } \hat{h}_Q^{-1}(t)$ (i.e. (4.2), (4.6) and (3.83)), we can suppose that

$$(q, t_0) \in B_P\left(\frac{r}{2}\right) \times (0, t_1].$$

By a similar argument in (3.15) and (3.16), and the definitions of T_S and T_Q , we have

$$\left(\Delta - \frac{\partial}{\partial t}\right) \text{tr } \hat{h}_S \geq c_3 |T_S(t)|_{H_S(t)}^2 - \tilde{c}_3 \quad (4.38)$$

and

$$\left(\Delta - \frac{\partial}{\partial t}\right) \text{tr } \hat{h}_Q^{-1} \geq c_4 |T_Q(t)|_{H_Q(t)}^2 - \tilde{c}_4 \quad (4.39)$$

on $B_P(2r)$, where c_3, \tilde{c}_3, c_4 and \tilde{c}_4 are constants depending only on $\max_{\overline{B_P(2r)}}(|F_{H_{0,S}}| + |F_{H_{0,Q}}|)$, the local C^0 bound of \hat{h}_S and \hat{h}_Q . Let $\tilde{c}_1 = \sup_{B_P(2r) \times [0, +\infty)} |G(t) + G_0|_{H(t)}^2$. Choosing $a \geq \frac{2c_1 + 8\tilde{c}_1 + 5Cr^{-2} + 1}{\min\{c_3, c_4, 2\}}$, and using (2.44), (4.31), (4.33), at the maximum point (q, t_0) , we have

$$0 \geq \left(\Delta - \frac{\partial}{\partial t}\right) \zeta_1 \geq |T_E|_{H_E(t)}^2 - c_5, \quad (4.40)$$

where constant c_5 depends only on the local C^0 bound of \tilde{h}_S and \tilde{h}_Q , r^{-2} , G_0 , $\max_{\overline{B_P(2r)}}(|F_{H_{0,S}}| + |F_{H_{0,Q}}|)$ and the lower bound of Ricci curvature of (M, ω) . We get a uniform constant c_6 such that

$$\sup_{(x,t) \in M \times [0, +\infty)} \zeta_1(x, t) \leq c_6, \quad (4.41)$$

and then

$$\begin{aligned} & \sup_{x \in B_P(\frac{r}{4}) \times [0, +\infty)} (|T_S(t)|_{H_S(t)}^2 + |T_Q(t)|_{H_Q(t)}^2 + |\gamma(t)|_{H_{S,Q}(t)}^2)(x, t) \\ & \leq \sup_{x \in B_P(\frac{r}{4}) \times [0, +\infty)} |T_E|_{H(t)}(x, t) \leq c_6. \end{aligned} \quad (4.42)$$

Since U is compact, by finite covering $\{B_{P_i}(r_i)\}_{i=1}^N$, there is a constant c_7 such that

$$\sup_{(x,t) \in U \times [0, +\infty)} (|T_S(t)|_{H_S(t)}^2 + |T_Q(t)|_{H_Q(t)}^2 + |\gamma(t)|_{H_{S,Q}(t)}^2)(x, t) \leq c_7. \quad (4.43)$$

□

5. LOCAL CURVATURE ESTIMATE

From (2.38), (4.18) and (4.19), we see

$$\begin{aligned} & \left(\Delta - \frac{\partial}{\partial t}\right) (|\gamma|_{H(t)}^2 + |T_S|_{H_S(t)}^2 + |T_Q|_{H_Q(t)}^2) \\ & \geq \frac{1}{4} (|F_{H_S}|_{H_S(t)}^2 + |F_{H_Q}|_{H_Q(t)}^2 + 2|\partial_{H(t)}\gamma|_{H(t)}^2) \\ & \quad - \hat{C}_1 |\gamma|_{H(t)}^2 (|\gamma|_{H(t)}^2 + |T_S|_{H_S(t)}^2 + |T_Q|_{H_Q(t)}^2) \\ & \quad - \hat{C}_2 (|\gamma|_{H(t)}^2 + |T_S|_{H_S(t)}^2 + |T_Q|_{H_S(t)}^2) - \hat{C}_3 \end{aligned} \quad (5.1)$$

on $M \setminus B_\Sigma(\delta)$, where constants \hat{C}_i depend only on the uniform local C^0 bound of \hat{h}_S and \hat{h}_Q , $H_{0,S}$, $H_{0,Q}$ and the lower bound of Ricci curvature of (M, ω) . Let's recall

$$|F_{H(t)}|_{H(t)}^2 = |F_{H_S(t)} - \gamma \wedge \gamma^*|_{H_S(t)}^2 + |F_{H_Q(t)} - \gamma^* \wedge \gamma|_{H_Q(t)}^2 + 2|\partial_{H(t)}\gamma|_{H(t)}^2, \quad (5.2)$$

and

$$\begin{aligned} & (\Delta - \frac{\partial}{\partial t})|F_{H(t)}|_{H(t)}^2 \\ & \geq 2|\nabla_{H(t)} F_{H(t)}|_{H(t)}^2 - \hat{C}_4(|F_{H(t)}|_{H(t)} + |Rm(\omega)|_\omega)|F_{H(t)}|_{H(t)}^2, \end{aligned} \quad (5.3)$$

where constant \hat{C}_4 depends only on the dimension of M . The proof of the inequality (5.3) can be found in Siu's lectures (page 31 in [42]).

Proof of Theorem 1.1 For simplicity, we denote

$$\nu = |\gamma|_{H(t)}^2 + |T_S|_{H_S(t)}^2 + |T_Q|_{H_Q(t)}^2. \quad (5.4)$$

By the estimate (4.20), we can choose a constant \hat{C}_5 such that

$$0 < \frac{1}{2}\hat{C}_5 \leq \hat{C}_5 - \nu(x, t) \leq \hat{C}_5 \quad (5.5)$$

for all $(x, t) \in M \setminus B_\Sigma(\delta) \times [0, +\infty)$. Let φ_3 and φ_4 be nonnegative cut-off functions satisfying

$$\varphi_3(x) = \begin{cases} 1, & x \in M \setminus B_\Sigma(4\delta), \\ 0, & x \in B_\Sigma(2\delta); \end{cases} \quad (5.6)$$

$$\varphi_4(x) = \begin{cases} 1, & x \in M \setminus B_\Sigma(2\delta), \\ 0, & x \in B_\Sigma(\delta); \end{cases} \quad (5.7)$$

and

$$|d\varphi_i|_\omega, |\Delta\varphi_i| \leq \frac{C}{r^2} \quad (5.8)$$

for all $i = 3, 4$.

Now we consider the following test function

$$\zeta_2 = \varphi_3^2 \frac{|F_{H(t)}|_{H(t)}^2}{\hat{C}_5 - \nu} + b\varphi_4\nu. \quad (5.9)$$

Let $\zeta_2(P, t_0) = \max_{M \times [0, t_1]} \zeta$, by the definition of the cut-off functions φ_i and the local uniform estimate (4.20), we can suppose that

$$(P, t_0) \in M \setminus B_\Sigma(2\delta) \times (0, t_1].$$

At the maximum point (P, t_0) , we have

$$\begin{aligned} & (\Delta - \frac{\partial}{\partial t})\zeta_2 \\ & = \varphi_3^2 \frac{1}{\hat{C}_5 - \nu} (\Delta - \frac{\partial}{\partial t})|F_{H(t)}|_{H(t)}^2 + \varphi_3^2 \frac{|F_{H(t)}|_{H(t)}^2}{(\hat{C}_5 - \nu)^2} (\Delta - \frac{\partial}{\partial t})\nu \\ & \quad - \varphi_3^2 \frac{2}{\hat{C}_5 - \nu} \nabla \left(\frac{|F_{H(t)}|_{H(t)}^2}{\hat{C}_5 - \nu} \right) \cdot \nabla (\hat{C}_5 - \nu) + b(\Delta - \frac{\partial}{\partial t})\nu \\ & \quad + \Delta \varphi_3^2 \frac{|F_{H(t)}|_{H(t)}^2}{\hat{C}_5 - \nu} + 2\nabla \varphi_3^2 \cdot \nabla \frac{|F_{H(t)}|_{H(t)}^2}{\hat{C}_5 - \nu} \end{aligned} \quad (5.10)$$

and

$$\nabla \left(\varphi_3^2 \frac{|F_{H(t)}|_{H(t)}^2}{\hat{C}_5 - \nu} \right) + b\nabla \nu = 0. \quad (5.11)$$

Putting (5.11) into (5.10), choosing the constants \hat{C}_5 and b large enough and using the formulas (5.1), (5.2), and (5.3), at the maximum point (p, t_0) , we have

$$\begin{aligned}
& (\Delta - \frac{\partial}{\partial t})\zeta_2 \\
&= \varphi_3^2 \frac{1}{\hat{C}_5 - \nu} (\Delta - \frac{\partial}{\partial t})|F_{H(t)}|_{H(t)}^2 + \varphi_3^2 \frac{|F_{H(t)}|_{H(t)}^2}{(\hat{C}_5 - \nu)^2} (\Delta - \frac{\partial}{\partial t})\nu \\
&\quad - \frac{2b}{\hat{C}_5 - \nu} |\nabla \nu|^2 + b(\Delta - \frac{\partial}{\partial t})\nu \\
&\quad + \Delta \varphi_3^2 \frac{|F_{H(t)}|_{H(t)}^2}{\hat{C}_5 - \nu} + \frac{2}{\hat{C}_5 - \nu} \nabla \varphi_3^2 \cdot \nabla |F_{H(t)}|_{H(t)}^2 \\
&\geq |F_{H(t)}|_{H(t)}^2 - \hat{C}_6,
\end{aligned} \tag{5.12}$$

where \hat{C}_6 is a positive constant depending only on the local uniform bound of ν , δ^{-1} and the curvature of (M, ω) . So we obtain

$$|F_{H(t)}|_{H(t)}^2(P, t_0) \leq \hat{C}_6. \tag{5.13}$$

Then there is a constant \hat{C}_7 such that

$$\sup_{M \setminus B_\Sigma(4\delta) \times [0, +\infty)} |F_{H(t)}|_{H(t)}^2 \leq \hat{C}_7. \tag{5.14}$$

This completes the proof of Theorem 1.1. □

REFERENCES

- [1] M. Atiyah and R. Bott, *The Yang-Mills equations over Riemann surfaces*, Phil. Trans. Roy. Soc. London A **308**(1982), 524-615.
- [2] L. Alvarez-Consul and O. Garcis-Prada, *Dimensional reduction, $SL(2, \mathbb{C})$ -equivariant bundles and stable holomorphic chains*, Int. J. Math., **2**(2001), 159-201.
- [3] O. Biquard, *On parabolic bundles over a complex surface*, J. London. Math. Soc., **53**(1996), no.2, 302-316.
- [4] S.B. Bradlow, *Vortices in holomorphic line bundles over closed Kähler manifolds*, Commun. Math. Phys. **135**(1990), 1-17.
- [5] S.B. Bradlow and O. Garcia-Prada, *Stable triples, equivariant bundles and dimensional reduction*, Math. Ann. **304** (1996), 225-252.
- [6] S. Bando and T. Mabuchi, *Uniqueness of Einstein Kähler metrics modulo connected group actions*, Algebraic geometry, Sendai, 1985, 11C40, Adv. Stud. Pure Math., 10, North-Holland, Amsterdam, 1987.
- [7] S. Bando and Y.T. Siu, *Stable sheaves and Einstein-Hermitian metrics*, in *Geometry and Analysis on Complex Manifolds*, World Scientific, 1994, 39-50.
- [8] P.D. Bartolomeis and G. Tian, *Stability of complex vector bundles*, J. Differential Geometry, **43** (1996), 232-275.
- [9] S.Y. Cheng and P. Li, *Heat kernel estimates and lower bound of eigenvalues*, Comment. Math. Helv. **56** (1981), 327-338.
- [10] T. Collins and A. Jacob, *Remarks on the Yang-Mills flow on a compact Kähler manifold*, Univ. Iagel. Acta Math. **51** (2013), 17-43.
- [11] G. Daskalopoulos, *The topology of the space of stable bundles on a Riemann surface*, J. Differential Geom. **36**(1992), 699-746.
- [12] G. Daskalopoulos and R. Wentworth, *Convergence properties of the Yang-Mills flow on Kähler surfaces*, J. Reine Angew. Math. **575**(2004), 69-99.
- [13] G. Daskalopoulos and R. Wentworth, *On the blow-up set of the Yang-Mills flow on Kähler surfaces*, Math. Z. **256**(2007), 301-310.

- [14] S.K.Donaldson, *A new proof of a theorem of Narasimhan and Seshadri*, J.Differential Geom., **18**(1983), 279-315.
- [15] S.K.Donaldson, *Anti-self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles*, Proc.London Math.Soc. **50**(1985), 1-26.
- [16] P.Griffiths and J.Harris, *Principles of algebraic geometry*, Pure and Applied Mathematics, Wiley-Interscience, New York, 1978.
- [17] A.Grigor'yan, *Gaussian upper bounds for the heat kernel on arbitrary manifolds*, J. Differential Geom. **45**(1997), no. 1, 33-52.
- [18] O.Garcia-Prada, *Dimensional reduction of stable bundles, vortices and stable pairs*, Int.J.Math. **5** (1994), 1-52.
- [19] H.Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero*, Ann. of Math. **79**(1964), no.1, 109-203.
- [20] H. Hironaka, *Flattening theorem in complex-analytic geometry*, Amer. J. Math. **97**(1975), no. 2, 503-547.
- [21] N.J.Hitchin, *The self-duality equations on a Riemann surface*, Proc.London Math.Soc. **55**(1987), 59-126.
- [22] D.Huybrechts and M.Lehn, *Stable pairs on curves and surfaces*, J. Algebraic Geom. **4**(1995), no. 1, 67-104.
- [23] M.C.Hong and G.Tian, *Asymptotical behaviour of the Yang-Mills flow and singular Yang-Mills connections*, Math. Ann. **330**(2004), no. 3, 441-472.
- [24] A.Jacob, *The Yang-Mills flow and the Atiyah-Bott formula on compact Kähler manifolds*, arXiv:1109.1550.
- [25] J.Jost and K.Zuo, *Harmonic maps and $Sl(r, \mathbb{C})$ -representations of fundamental groups of quasiprojective manifolds*, J. Algebraic Geom. **5**(1996), no. 1, 77-106.
- [26] J.Y.Li, *Hermitian-Einstein metrics and Chern number inequalities on parabolic stable bundles over Kähler manifolds*, Comm. Anal. Geom. **8**(2000), no. 3, 445-475.
- [27] J.Y.Li and M.S.Narasimhan, *Hermitian-Einstein metrics on parabolic stable bundles*, Acta Math. Sin. (Engl. Ser.) **15** (1999), no. 1, 93-114.
- [28] J.Y.Li and X.Zhang, *The gradient flow of Higgs pairs*, J. Eur. Math. Soc., **13**(2011), 1373-1422.
- [29] J.Y.Li and X.Zhang, *Existence of approximate Hermitian-Einstein structures on semi-stable Higgs bundles*, Calc. Var. Partial Differential Equations, **52**(2015), no. 3-4, 783-795.
- [30] J.Y.Li, C.J.Zhang and X.Zhang, *Semi-stable Higgs sheaves and Bogomolov type inequality*, arXiv:1601.00729.
- [31] J.Li and S.T.Yau, *Hermitian-Yang-Mills connection on non-Kähler manifolds*, Mathematical aspects of string theory (San Diego, Calif., 1986), 560-573, Adv. Ser. Math. Phys., 1, World Sci. Publishing, Singapore, 1987.
- [32] P.Li, *On the Sobolev constant and the p -spectrum of a compact Riemannian manifold*, Ann. Sci. Ecole Norm. Sup. **13**(1980), no. 4, 451-468.
- [33] P.Li, *Geometry analysis*, Cambridge Studies in Advanced Mathematics (No. 134).
- [34] P.Li and L.F.Tam, *The heat equation and harmonic maps of complete manifold*, Invent. Math. **105**(1991), 1-46.
- [35] T.Mochizuki, *Kobayashi-Hitchin correspondence for tame harmonic bundles and an application*, Astérisque **309** (2006), ISBN: 978-2-85629-226-6, +117pp.
- [36] T.Mochizuki, *Kobayashi-Hitchin correspondence for tame harmonic bundles*, Geom. Topol. **13**(2009), no. 1, 359-455.
- [37] I. Mundet i Riera, *A Hitchin-Kobayashi correspondence for Kähler fibrations*, J.reine angew. Math. **528**(2000), 41-80.
- [38] M.S.Narasimhan and C.S.Seshadri, *Stable and unitary vector bundles on compact Riemann surfaces*, Ann. of Math., **82** (1965) 540-567.
- [39] B.Sibley, *Asymptotics of the Yang-Mills flow for holomorphic vector bundles over Kähler manifolds: the canonical structure of the limit*, J. Reine Angew. Math. **706**(2015), 123-191.
- [40] B.Sibley and R.Wentworth, *Analytic cycles, Bott-Chern forms, and singular sets for the Yang-Mills flow on Kähler manifolds*, Advances in Mathematics, **279**(2015), 501-531.
- [41] C.T.Simpson, *Constructing variations of Hodge structures using Yang-Mills connections and applications to uniformization*, J.Amer.Math.Soc., **1**, (1988), 867-918.
- [42] Y.T.Siu, *Lectures on Hermitian-Einstein metrics for stable bundles and Kähler-Einstein metrics*, Birkhauser, Basel-Boston, (1987), MR **89d**:32020.
- [43] K.K. Uhlenbeck, *Connections with L^p bounds on curvature*, **83**(1) (1982), Comm. Math. Phys. 31-42.
- [44] K.K. Uhlenbeck, *A priori estimates for Yang-Mills fields*, unpublished manuscript.
- [45] K.K.Uhlenbeck and S.T.Yau, *On existence of Hermitian-Yang-Mills connection in stable vector bundles*, Comm.Pure Appl.Math., **39S**(1986), 257-293.
- [46] C. Voisin, *Hodge Theory and complex algebraic geometry*, Cambridge University Press 2002.
- [47] G.Wilkin, *Morse theory for the space of Higgs bundles*, Comm. Anal. Geom., **16**(2)(2008), 283-332.

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